Calculus of Variations, Lecture 3, 2/6/2017

[start with the example at end of Lecture 2 notes, concerning lower bds on 1st Dirichlet eigenvalue of a domain in $\mathbb{R}^n$.]

Today:

a) another example of duality, this time involving L^1 and L^∞ type norms

b) an example of "the calibration method" in the calculus of variations (not exactly convex duality, but closely related in concept)

Starting pt for our L^1-L^∞ example: what can we say about

$$(*) \qquad \min_{\sigma} \{ \|\sigma\|_\infty \text{ s.t. } \operatorname{div} \sigma = 1 \text{ on } \Omega \}$$

where (to fix ideas) Ω is a bounded domain in $\mathbb{R}^n$?

Interpretation: it's raining uniformly on Ω. How can rain flow to bdry with least possible local accumulation? (Note:

continuum pblm has a natural analogue on any graph. Arguments presented below also have analogues in graph setting.)

Observation: it is equivalent to solve

$$(**) \quad \max_{\substack{|\sigma| \le 1 \\ \operatorname{div} \sigma = \lambda \text{ (constant)}}} \lambda$$

since if λ_{max} is optimal for $(**)$ then

$$\left.\begin{array}{l} \operatorname{div} \sigma = \lambda \text{ (const)} \\ |\sigma| \le 1 \end{array}\right\} \Rightarrow \lambda \le \lambda_{max}$$

so

$$\left.\begin{array}{l} \operatorname{div}\left(\frac{1}{\lambda}\sigma\right) = 1 \\ \left|\frac{1}{\lambda}\sigma\right| \le 1/\lambda \end{array}\right\} \Rightarrow \frac{1}{\lambda} \le \frac{1}{\lambda_{max}}.$$

Thus $1/\lambda_{max}$ is the optimal value for $(*)$.

We identify a dual pblm by the min/max procedure discussed in Lecture 2: $(**)$ is equiv to

(P) $$\max_{\substack{|\sigma|\le 1 \\ \operatorname{div}\sigma=\lambda}} \lambda = \max_{|\sigma|\le 1} \min_{\substack{u=0 \text{ at } \partial\Omega \\ \int_\Omega u\,dx=1}} -\int \langle \sigma, \nabla u\rangle$$

(since min over u is $-\infty$ unless $\operatorname{div}\sigma = \lambda$ is constant, in which case it equals λ).

Assuming max min = min max, dual is

$$\min_{\substack{u=0 \text{ at } \partial\Omega \\ \int_\Omega u=1}} \max_{|\sigma|\le 1} -\int \langle \sigma, \nabla u\rangle$$

Best σ has $-\langle \sigma, \nabla u\rangle = |\nabla u|$, so dual is

(D) $$\min_{\substack{u=0 \text{ at } \partial\Omega \\ \int_\Omega u\,dx=1}} \int_\Omega |\nabla u|\,dx$$

Is max min = min max? An inequality is elementary, as explained in Lecture 2; in this case

$$|\sigma|\le 1,\ \operatorname{div}\sigma=\lambda \text{ (constant)} \Rightarrow -\int_\Omega \langle\sigma,\nabla u\rangle \le \int_\Omega |\nabla u|$$
$$u=0 \text{ at } \partial\Omega,\ \int_\Omega u\,dx=1$$
$$\Rightarrow \lambda \le \int_\Omega |\nabla u| .$$

so $\max \mathcal{P} \le \min \mathcal{D}$. (In this example the "primal" is a concave maximization & the "dual" is a convex minimization.)

As for equality ($\max\min = \min\max$): this is not simple to prove in this case. I'll distribute a separate set of notes with a self-contained proof and some refs to relevant literature.

Another (actually, related) feature of this problem: the optimal u is rather singular — in fact, it's the char fn of a set (to be explained below).

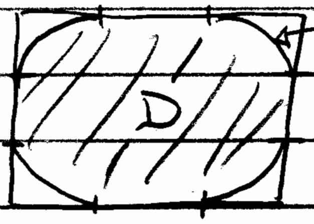
Comment: this example captures an interesting feature of L^1-L^∞ duality pairs: one problem is typically much easier to solve (perhaps explicitly) than the other.

In the present setting we can solve $\mathcal{D}$ more or less explicitly, using the characterization

$$(***) \qquad \min_{\substack{u=0 \text{ at } \partial\Omega \\ \int_\Omega u\,dx = 1}} \int_\Omega |\nabla u| = \min_{D \subset \Omega} \frac{\text{length}(\partial D)}{\text{Area}(D)}$$

= solution of a sort of "isoperimetric" problem!

For example, if Ω = square,



(see G. Strang, "A minimax problem in plasticity theory", Springer Lecture Notes in Math 701, 1979, 319-333).

Let me sketch the proof of (***). A key ingredient is the

<u>Coarea Formula</u> $\int f(x)|\nabla u|\,dx = \int \left(\int_{u=t} f\,ds \right) dt$

(this is easy to justify if u is nice enough – more or less it is just the "method of shells" from Calc III – but with slightly more careful notation it extends to any $u \in BV$).

Now proceeding in steps:

<u>step 1</u>: LHS of (***) $= \min\limits_{u=0 \text{ at } \partial\Omega} \dfrac{\int_\Omega |\nabla u|}{\int_\Omega u\,dx}$

(easy: if $\int_\Omega u\,dx = c \neq 0$ RHS is unchanged when we replace u by u/c.)

<u>step 2</u> May suppose $u \geq 0$, since replacing u by $|u|$ leaves $\int |\nabla u|$ unchanged and it increases $\int_\Omega u\,dx$.

<u>step 3</u> For $u \geq 0$, $\int_\Omega u\,dx = \int_0^\infty \text{Area}\{u \geq t\}\,dt$

since
$$\int_\Omega u\,dx = \int_\Omega \int_0^{u(x)} 1\,dt\,dx$$

(now use Fubini's Thm). On the other hand
$$\int_\Omega |\nabla u|\,dx = \int_0^\infty \text{length}\{u = t\}\,dt.$$

(by Co-area formula with $f = 1$). So: if RHS of (***) has min $= \alpha$ then

$$\text{length}\{u = t\} \geq \alpha\,\text{Area}\{u \geq t\}.$$

for all t (by taking $D = \{u \geq t\}$).

So $\int_\Omega |\nabla u| \geq \alpha \int_\Omega u\, dx$ for any function u s.t. $u \geq 0$ and $u = 0$ at $\partial\Omega$

This shows

$$\min_{\substack{u=0 \text{ at } \partial\Omega \\ u \geq 0}} \frac{\int_\Omega |\nabla u|}{\int_\Omega u} \geq \min_{D \subset \Omega} \frac{\text{length}(\partial D)}{\text{Area}(D)}.$$

But the opposite inequality is easy (just take u = "char fn of" D).

<u>Digression</u>: For discn of closely related probs see G. Strang, "Maximum flows and minimum cuts in the plane", J Global Optzn 47 (2010) 527-535. One of the many topics there is a very efficient proof (due to Grieser) of

<u>Cheeger's inequality</u>: if λ_0 = 1st Dirichlet eigenvalue of Δ in Ω, and

$$h = \min_{D \subset \Omega} \frac{\text{length}(\partial D)}{\text{area}(D)}$$

Then

$$\frac{h^2}{4} \leq \lambda_0.$$

Pf: From prior discussion $\exists\, \sigma$ st $|\sigma| \le 1$ and $\operatorname{div} \sigma = h$. Let u_0 be the 1st Dirichlet eigenfunction. Then

$$h \int_\Omega u_0^2 = \int_\Omega (\operatorname{div} \sigma)\, u_0^2 = -2 \int_\Omega u_0 \langle \sigma, \nabla u_0 \rangle$$

$$\le 2 \int_\Omega |u_0|\, |\nabla u_0|$$

$$\le 2 \left(\int_\Omega u_0^2\right)^{1/2} \left(\int_\Omega |\nabla u_0|^2\right)^{1/2}$$

so

$$h \le \frac{2\left(\int |\nabla u_0|^2\right)^{1/2}}{\left(\int u_0^2\right)^{1/2}} = 2 \lambda_0^{1/2}$$

Our example of "the calibration method" follows the paper by G. De Philippis + E. Paolini, "A short proof of the minimality of the Simons cone," Rend. Sem. Mat. Univ Padova 121 (2009) 233-241.

First, some orientation: essence of duality is a scheme for proving lower bds on minimization problems (or upper bds on maximization problems) using just integration by parts + elementary inequalities (applied to a well-chosen test function).

Essence of "calibration method": sometimes this works even for problems that are not (in any obvious way) convex.

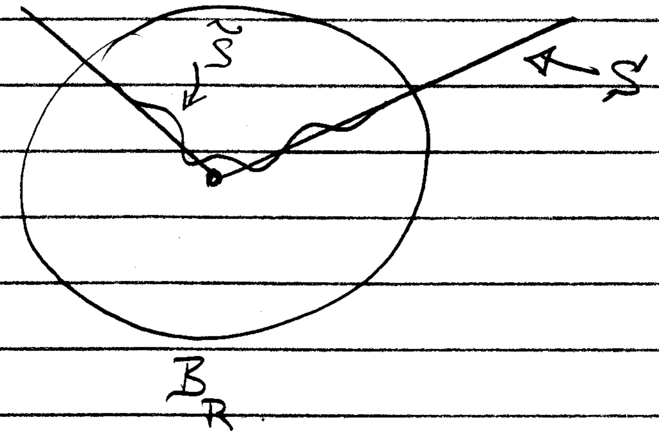
The example I'll discuss comes from the theory of minimal surfaces: the 7-dimensional surface in $\mathbb{R}^8$

$$S = \{ x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2 \}$$

(known as the Simons cone) is area-minimizing in the sense that no compactly-supported perturbation lowers the (7-dimensional) surface area.

Rough sketch:

$S \cap B_R$ has min poss area among all $\tilde{S}$ st $\tilde{S}|_{\partial B_R} = S|_{\partial B_R}$.



Importance of this: codim-one minimal surfaces are smooth in $\mathbb{R}^n$, $n \le 7$. This example shows that $n=8$ is different. Original pf (much more complicated!) was due to Bombieri, De Giorgi, + Giusti in 1969.

<u>Key idea</u>: it's suff't to find a vector field σ in $\mathbb{R}^8$ s.t.

$|\sigma| = 1$ ptwise

σ = unit normal at all pts on the Simons cone

$\operatorname{div} \sigma \le 0$ "below" the Simons cone

$\operatorname{div} \sigma \ge 0$ "above" the Simons cone

("above" means $x_1^2 + x_2^2 + x_3^2 + x_4^2 > x_5^2 + x_6^2 + x_7^2 + x_8^2$, and "below" means the opposite inequality).

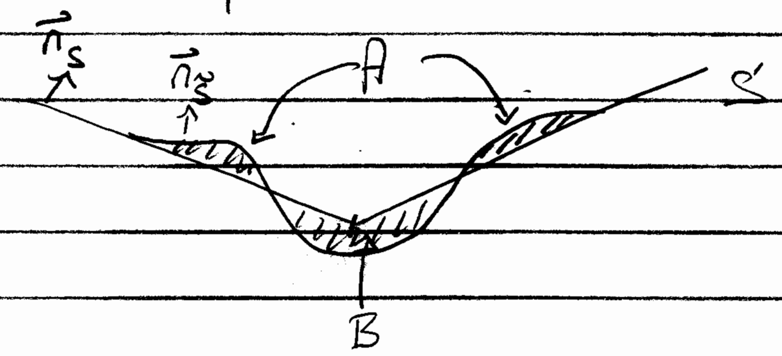
<u>Claim</u>: Any such σ provides an elementary, integration-by-parts-based proof that (in the notation of our previous sketch) $|\tilde{S}| \le |S|$

(Analogy to convex duality: while σ does not solve a "dual pbm", it provides a bound by arguments similar to those used in using test fns from a dual pbm to bound the primal.)

<u>Proof of the claim:</u> given $\tilde{S}$, let

$$A = \{\text{pts below } \tilde{S} + \text{above } S\}$$

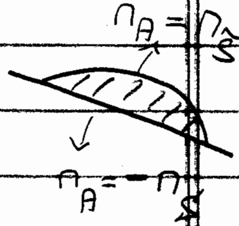
$$B = \{\text{pts below } S + \text{above } \tilde{S}\}.$$



and let $\vec{n}_S$, $\vec{n}_{\tilde{S}}$ be unit normals pointing "upward".

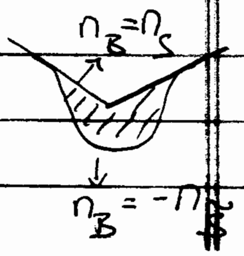
Observe that

$$\int_A \operatorname{div} \sigma = \int_{\partial A} \sigma \cdot \vec{n}_A \quad \text{using the outward unit normal } \vec{n}_A$$



$$= \int_{(\partial A)\cap \tilde{S}} \sigma \cdot n_{\tilde{S}} - \int_{(\partial A)\cap S} \sigma \cdot n_S$$

$$\int_B \operatorname{div} \sigma = \int_{\partial B} \sigma \cdot \vec{n}_B \quad \text{using outward unit normal } \vec{n}_B$$



$$= \int_{(\partial B)\cap S} \sigma \cdot n_S - \int_{(\partial B)\cap \tilde{S}} \sigma \cdot n_{\tilde{S}}$$

Therefore

$$\int_A \operatorname{div}\sigma - \int_B \operatorname{div}\sigma = \int_{\tilde{S}\cap B_R} \sigma\cdot n_{\tilde{S}} - \int_{S\cap B_R} \sigma\cdot n_S$$

(pts where $S=\tilde{S}$ aren't in either ∂A or ∂B, but such points enter into both terms in RHS and the contributions cancel).

<u>Note:</u> if σ has properties listed above then LHS terms are positive, so

$$0 \le \int_A \operatorname{div}\sigma - \int_B \operatorname{div}\sigma$$

$$= \underbrace{\int_{\tilde{S}\cap B_R} \sigma\cdot n_{\tilde{S}}}_{\substack{\text{This is}\\ \le \text{ area of } \tilde{S}\\ \text{in } B_R \text{, since}\\ |\sigma|\le 1}} - \underbrace{\int_{S\cap B_R} \sigma\cdot n_S}_{\substack{\text{This is exactly area}\\ \text{of } S \text{ in } B_R\text{, since}\\ \sigma = n_S \text{ on } S}}$$

So area of S in B_R $\le$ area of $\tilde{S}$ in B_R

as asserted.

<u>Where to find σ</u>? It's easy: writing

let $z = (x_1, x_2, x_3, x_4)$, $w = (x_5, x_6, x_7, x_8)$,

$$f(z,w) = \frac{|z|^4 - |w|^4}{4}$$

and consider

$$\sigma = \frac{\nabla f}{|\nabla f|}.$$

(Note: "above S" $\iff$ $f > 0$.)

An elementary (dimension-dependent!) calcu. gives

$$|\nabla f|^3 \operatorname{div} \frac{\nabla f}{|\nabla f|} = (|z|^4 - |w|^4) \underbrace{\left[3|z|^4 - 6|z|^2|w|^2 + 3|w|^2\right]}_{\text{clearly} \geq 0}$$

so $\operatorname{div}\sigma$ has same sign as f (as desired). And Simons cone is $\{f = 0\}$, so $\sigma = n_S$ on S as required.

<u>Some exercises:</u>

(1) Show that the problems

$$(\mathcal{P}) \qquad \min_{\substack{\operatorname{div}\sigma = F \text{ in } \Omega \\ \sigma \cdot n = f \text{ at } \partial\Omega}} \int_\Omega |\sigma|$$

and

$$(\mathcal{D}) \qquad \max_{|\nabla u|_{L^\infty} \le 1} \int_{\partial\Omega} u \cdot f \, ds - \int_\Omega u \cdot F \, dx$$

are a dual pair, if $\int_\Omega F\,dx = \int_{\partial\Omega} f\,ds$.

How should σ + ∇u be related if equality is to hold?

(Hint: $\max_{\sigma \in \mathbb{R}^n} \langle \xi, \sigma \rangle - |\sigma| = \begin{cases} 0 & \text{if } |\xi| \le 1 \\ \infty & \text{if } |\xi| > 1 \end{cases}$.)

Rmk: if $\Omega \subset \mathbb{R}^2$ and $F = 0$ then $\mathcal{P}$ can be solved explicitly in simple cases using the co-area formula. Why?

(2) The L^1-L^∞ example discussed in the 1st half of these notes concerned

$$\min_\sigma \{ \|\sigma\|_\infty \text{ st } \operatorname{div} \sigma = 1 \text{ on } \Omega \}.$$

Can you do something similar for

$$\min_\sigma \left\{ \max_{x \in \Omega} \left(|\sigma_1(x)| + |\sigma_2(x)| \right) \text{ st } \operatorname{div}\sigma = 1 \text{ in } \Omega \right\}$$

where $\Omega \subset \mathbb{R}^2$?