Addendum to Lecture 3: why is
min max = max min in our L^1-L^∞ example?

The paper

R. Temam, "Mathematical problems in plasticity theory," in Variational Inequalities and Complementarity Problems: Theory + Applications, RW Cottle et al eds, John Wiley + Sons, 1979 [pp 357-373]

shows how this problem can be cast as a special case of a duality theorem in the book by Ekeland + Temam, Convex Analysis + Variational Problems, North Holland, 1976 (which is on reserve).

The following self-contained treatment is a little different - essentially a specialization of the argument given for a class of problems from plasticity in E. Christiansen, "Limit analysis in plasticity as a mathematical programming problem," Calcolo 17 (1980) 41-65

Goal: given a bdd domain $D \subset \mathbb{R}^n$ (with smooth enough bdry) and $f : D \to \mathbb{R}$ (suff'tly regular, see below), choose function spaces X + Y (natural to the problem) such that

$$\sup_{\substack{\|\sigma\|_{L^\infty} \le 1 \\ \sigma \in X}} \inf_{\substack{\int_D uf = 1 \\ u \in Y}} \int \langle \sigma, \nabla u \rangle = \inf_{\substack{\int_D uf=1 \\ u \in Y}} \sup_{\substack{\|\sigma\|_{L^\infty} \le 1 \\ \sigma \in X}} \int \langle \sigma, \nabla u \rangle$$

(Our L^1-L^∞ example in Lecture 3 used $f = 1$.)

<u>Choices</u>:

a) $X = W^{1,p}(D)$, with $p > n$ (so that $\sigma \in X \Rightarrow$ σ is continuous)

b) $Y = BV(\mathbb{R}^n) \cap \{u = 0 \text{ outside } D\}$. While a naive formulation would ask that $u = 0$ at ∂D, we have to allow u to jump at the boundary (indeed, we saw using the co-area formula that the actual solution <u>has</u> this character). By defn, $u \in BV \Leftrightarrow$ ∇u is a vector-valued measure with finite total variation $\int |\nabla u|$. Fact: $\|u\|_{L^{n/n-1}(D)} \le C \int |\nabla u|$.

c) Since u may jump to 0 at ∂D, ∇u can "charge" ∂D, so we must define

$$\int \langle \sigma, \nabla u \rangle = \int_{\mathbb{R}^n} \langle \sigma, \nabla u \rangle = \int_{\overline{D}} \langle \sigma, \nabla u \rangle$$

which is not the same in general as $\int_D \langle \sigma, \nabla u \rangle$

With these choices it is easy to see that

The sup-inf is $\sup\left\{\lambda : \exists\, \sigma \in W^{1,p},\ -\operatorname{div}\sigma = \lambda f,\ |\sigma| \le 1 \text{ ptwise}\right\}$

the inf-sup is $\displaystyle\inf_{\substack{\int_D uf = 1 \\ u=0 \text{ outside } D}} \int |\nabla u|$

(the arguments are exactly as we did in class for $f=1$.)
To be sure the sup-inf is positive it is natural to assume that

d) there exists $\tau \in W^{1,p}(D)$ with $\operatorname{div}\tau = f$. (This is effectively a regularity hypothesis on f; by elliptic regularity it suffices that $f \in L^p$. We can then solve $\Delta\varphi = f$ in D, $\varphi = 0$ at ∂D, and take $\tau = \nabla\varphi$. Since elliptic theory $\Rightarrow \|\varphi\|_{W^{2,p}(D)} \le C\|f\|_{L^p(D)}$, this τ is in $W^{1,p}(D)$.)

Since sup inf $\le$ inf sup trivially, our task is to show that inf sup $\le$ sup inf, ie that

$$\inf_{\substack{\int_D uf = 1 \\ u=0 \text{ outside } D}} \int |\nabla u| \le \sup\left\{\lambda : \exists\, \sigma \in W^{1,p},\ -\operatorname{div}\sigma = \lambda f,\ |\sigma| \le 1 \text{ ptwise}\right\}.$$

It's convenient to define

$$F(\sigma) = \inf_{\substack{\int uf = 1 \\ u \in Y}} \int \langle \sigma, \nabla u \rangle = \begin{cases} \lambda & \text{if } -\operatorname{div} \sigma = \lambda f \\ -\infty & \text{otherwise} \end{cases}$$

and to observe that since F is an inf of linear functionals, it is a <u>concave</u> function of σ.

Now we start the real work. Let μ be the value of the sup inf (ie $\mu = \sup_{\substack{\sigma \in X \\ \|\sigma\|_{L^\infty} \le 1}} F(\sigma)$), and consider

$$S_1 = \{ (\sigma, r) \in X \times \mathbb{R} : F(\sigma) - \mu \ge r \}$$

$$S_2 = \{ (\sigma, r) \in X \times \mathbb{R} : \|\sigma\|_{L^\infty} \le 1, \ r \ge 0 \}.$$

Then

- S_1 + S_2 are both convex (we use here the concavity of F)
- S_2 has nonempty interior (in fact $\sigma = 0, r = 1$ is an interior point; we use here that $p > n$ so that $\|\sigma\|_{L^\infty} \le C \|\sigma\|_{W^{1,p}}$).

Therefore there is a linear functional on $X \times \mathbb{R}$ that "separates" S_1 + S_2 (this is a corollary of the Hahn-Banach theorem, see eg Royden [in my copy = 2nd edn it is Thm 20 in Chap 10]). This means $\exists\, \ell \in (W^{1,p})^*$ and constants $\bar{r}, c \in \mathbb{R}$ st

(i) $\ell(\sigma) + r\bar{r} \geq c$ for all $(\sigma, r) \in S_1$

(ii) $\ell(\sigma) + r\bar{r} \leq c$ for all $(\sigma, r) \in S_2$

<u>Claim</u>: $\bar{r} < 0$. In fact, $(0, r) \in S_2$ for all $r > 0$; substitution into (ii) shows (as $r \to \infty$) that $\bar{r} \leq 0$.

Can $\bar{r} = 0$? If so then since $(0, -\mu) \in S_1$, (i) forces $c \leq 0$. But since $(\sigma, 0) \in S_2$ when $\|\sigma\|_X$ is suffly small, (ii) forces $c > 0$. This is a contradiction.

So $\bar{r} < 0$, + the claim is proved.

Rescaling ℓ + c, we may suppose $\bar{r} = -1$; then (i) + (ii) become

(i) $\ell(\sigma) - r \geq c$ for $(\sigma, r) \in S_1$

(ii) $\ell(\sigma) - r \leq c$ for $(\sigma, r) \in S_2$.

We're almost done. Recall (from bottom of pg 3) that our task is to show

$$\inf_{\substack{\int uf = 1 \\ u \in Y}} \int |\nabla u| \leq \mu .$$

We do this by showing

<u>Claim 1</u>: $\displaystyle \sup_{\substack{\|\sigma\|_{L^\infty} \leq 1 \\ \sigma \in X}} \ell(\sigma) = c \leq \mu$

<u>Claim 2</u>: $\ell(\sigma) = \int \langle \sigma, \nabla u_0 \rangle$ for some $u_0 \in Y$

<u>Claim 3</u>: $\int u_0 f = 1$

These suffice, since

$$\sup_{\substack{\|\sigma\|_{L^\infty} \leq 1 \\ \sigma \in X}} \int \langle \sigma, \nabla u_0 \rangle = \int |\nabla u_0|$$

<u>Proof of Claim 1</u>: Observe first that since $(0, -\mu) \in S_1$, we have $c \leq \mu$ from (i).

Now observe that $\exists\ \sigma_j \in X$, $\|\sigma_j\|_{L^\infty} \le 1$ s.t. $F(\sigma_j) \uparrow \mu$. Setting $r_j = F(\sigma_j) - \mu$ we have

$$\ell(\sigma_j) - r_j \geq c \quad \text{by (i)}$$
$$\ell(\sigma_j) \leq c \quad \text{by (ii)}$$

whence $\ell(\sigma_j) \to c$. This shows

$$\sup_{\substack{\|\sigma\|_{L^\infty} \le 1 \\ \sigma \in X}} \ell(\sigma) \geq c\,,$$

but the opposite inequality is obvious from (ii). So

$$\sup_{\substack{\|\sigma\|_{L^\infty} \le 1 \\ \sigma \in X}} \ell(\sigma) = c$$

and Claim 1 is proved.

<u>Proof of Claim 2</u>: since $\sup\limits_{\|\sigma\|_{L^\infty} \le 1} \ell(\sigma)$ is bounded, there is a vector-valued measure $\vec{\mu} = \{\mu_i\}$ with finite total variation such that

$$\ell(\sigma) = \int_{\overline{D}} \sum \sigma_i \, d\mu_i\,.$$

Our task is to show that $\vec{\mu} = \nabla u$ for some $u \in \mathcal{Y}$.

A key observation is that

$$\ell(\sigma) = 0 \quad \text{whenever} \quad \operatorname{div} \sigma = 0$$

Indeed, if $\operatorname{div}\sigma = 0$ then $F(\sigma) = 0$ and moreover $F(t\sigma) = 0$ for any $t \in \mathbb{R}$. So

$$\begin{aligned} \operatorname{div}\sigma = 0 &\Rightarrow (t\sigma, -\mu) \in S_1 \text{ for all } t \\ &\Rightarrow t\,\ell(\sigma) + \mu \geq c \text{ for all } t \\ &\Rightarrow \ell(\sigma) = 0. \end{aligned}$$

Thus ℓ can be viewed as a linear functional on $W^{1,p}(D)/\{\sigma : \operatorname{div}\sigma = 0\}$. But this quotient is isomorphic to L^p, via the map

$$T : \sigma \longmapsto \operatorname{div}\sigma$$

(This follows from the Closed Graph Theorem once we recognize that T is onto. To see that, note that for $g \in L^p(D)$, the solution of $\Delta\varphi = g$ in D, $\varphi = 0$ at ∂D has $\|\varphi\|_{W^{2,p}} \leq C\|g\|_{L^p}$ so that $\sigma = \nabla\varphi \in W^{1,p}$ has $T(\sigma) = g$.)

Since $(L^p)^* = L^q$, $\frac{1}{p} + \frac{1}{q} = 1$, $\exists\, u_0 \in L^q(D)$ s.t.

$$\ell(\sigma) = -\int (\operatorname{div}\sigma)\, u_0$$

Extending u_0 by 0 outside D, we can write

Thus as

$$\ell(\sigma) = \int \langle \sigma, \nabla u_0 \rangle$$

Evidently the measure $\bar{\mu}$ introduced when we started this proof is really $\bar{\mu} = \nabla u_0$. We have $u_0 \in Y$ since

$$\int |\nabla u_0| = \sup_{\|\sigma\|_{L^\infty} \leq 1} \int \langle \sigma, \nabla u_0 \rangle$$

is known to be finite. Claim 2 is proved.

<u>Proof of Claim 3</u>: Let $\tau \in W^{1,p}$ satisfy $-\operatorname{div} \tau = f$. Then

$$F(\tau) = 1 ,$$

so

$$(\lambda \tau, \lambda - \mu) \in S_1 \quad \text{for any } \lambda \in \mathbb{R} ,$$

Therefore

$$\lambda \ell(\tau) - \lambda + \mu \geq c \quad \text{for any } \lambda \in \mathbb{R} ,$$

which can hold only if $\ell(\tau) = 1$. Since $\ell(\sigma) = \int \langle \sigma, \nabla u_0 \rangle$ we have

$$1 = \int \langle \tau, \nabla u_0 \rangle = \int -\operatorname{div} \tau \cdot u_0 = \int f u_0 .$$

Claim 3 is now complete.

The preceding argument is special in its choice of function spaces, but typical in the sense that the "separating hyperplane theorem" (or some other version of the Hahn-Banach theorem) underlies most proofs of duality, other than those done by explicit examination of the solutions (as we did for quadratic cases).

It is natural to ask whether sup inf = inf sup can fail. The article by Christiansen gives an example where the failure is due to a poor choice of function spaces, namely restriction of both σ + u to be C^k functions. In fact

$$\inf_{\substack{u\in C^k \\ u=0 \text{ at } \partial D \\ \int uf=1}} \ \sup_{\substack{\sigma\in C^k \\ |\sigma|\le 1}} \int_D \langle \nabla u, \sigma \rangle = \inf_{\substack{u\in C^k \\ u=0 \text{ at } \partial D \\ \int uf=1}} \int_D |\nabla u|$$

is strictly positive (in fact $\geq \mu$, since the inf is being taken over a subset of Y). However we have

$$\inf_{\substack{u \in C^k \\ u=0 \text{ at } \partial D \\ \int uf = 1}} \int \langle \sigma, \nabla u \rangle = \begin{cases} \lambda & \text{if } -\operatorname{div}\sigma = \lambda f \\ -\infty & \text{otherwise} \end{cases}$$

from which it follows that if $f \notin C^{k-1}$ then

$$\sup_{\substack{\sigma \in C^k \\ |\sigma| \leq 1}} \inf_{\substack{u \in C^k \\ u=0 \text{ at } \partial D \\ \int uf = 1}} \int \langle \sigma, \nabla u \rangle = 0 .$$

For other examples where sup inf $\neq$ inf sup in problems closely related to our $L^1 - L^\infty$ example, see R. Nozawa, "Examples of max-flow and min-cut problems with duality gaps in continuous networks", Math Prog 63 (1994) 213-234