Calculus of Variations, Lecture 12, 4/24/2017

Topic of these notes: the "Modica-Mortola problem", which (besides much intrinsic interest) provides a convenient introduction to Γ-convergence.

Some orientation first:

- We know that some var'l pbms require min sequences to be highly oscillatory, eg

$$\int_0^1 (u_x^2-1)^2 + \lambda u^2 \, dx$$

while others permit oscillatory behavior without requiring it, eg

$$\int_0^1 (u_x^2-1) \, dx$$

In either case, a higher-order regularization (with a small coefficient) selects the scale + local character of minimizers, eg

$$\int_0^1 (u_x^2-1)^2 + \varepsilon^2 u_{xx}^2 + \lambda u^2$$

or (equivalently, up to relations btwn λ + ε)

$$\int \frac{1}{\varepsilon}(u_x^2-1)^2 + \varepsilon u_{xx}^2 + \lambda u^2$$

- More physical examples with character similar to last bullet include

  - multiwell energies modeling mixtures of martensite variants (regularizing term here is "surface energy" of phase interfaces)
  - membrane energy of a thin elastic sheet (regularizing term here is "bending energy").

- A more familiar example of a family of var'l plms with a small parameter is

$$(\Delta x)\sum' \frac{|u_i - u_{i+1}|^2}{(\Delta x)^2} \underset{\Delta x \to 0}{\longrightarrow} \int u_x^2$$

This is much tamer, since as $\Delta x \to 0$ we don't expect oscillatory behavior (minimizers converge strongly in $W^{1,p}$ — actually of course they are essentially linear fns)

Numerical analysis of var'l plms is about design + analysis of schemes that avoid osc behavior as $\Delta x \to 0$ (starting, usually, from a well-posed cont's var'l plm).

Γ-convergence is <u>different</u>: focus is on var'l pbms (often from physics or mechanics) where min sequences develop structure at a scale that tends to 0 with ε. Goal is to understand the preferred behavior + identify a limiting var'l pbm that incorporates this understanding. (Note the strong parallel to our discn of "relaxation".)

<u>Modica - Mortola pbm</u> is just one example, but a convenient one since it is both important + relatively easy. Some refs:

- Relatively easy reading, + basis of my discn of "local minimizers" is RV Kohn + P Sternberg, "Local minimizers + singular perturbations", Proc Roy Soc Edinburgh 111A (1989) 69-84

- Early articles were: L. Modica + S. Mortola, Boll. Unione Mat. Italiana, 2 articles in Italian in 1977 (no vol constraint); then P. Sternberg, ARMA 101 (1988) 209-260 + L. Modica, ARMA 98 (1987) 123-142 (with vol constraint)

- For broader perspective on Γ-convergence see A. Braides' book "Γ-convergence for beginners"

(OUP, avail thru Bobcat as an ebook, easy to read but limited to 1D problems) or A. Braides notes "A handbook of Γ-convergence" (avail at axp.mat.uniroma2.it/~braides/Handbook.pdf, publ in "Handbook of Diff'l Eqns: Stationary Partial Diff'l Eqns, Vol 3," Chipot + Quittner eds, Elsevier, 2006).

Getting started: our goal is to "understand asymptotic behavior of"

$$F_\varepsilon(u) = \int_\Omega \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon}(u^2-1)^2 \, dx$$

in limit as $\varepsilon \to 0$ (here $\Omega \subset \mathbb{R}^n$ is bounded + $u: \Omega \to \mathbb{R}$). Ans will be: F_ε "Γ-converges" to

$$F_0(u) = \begin{cases} \frac{8}{3} \operatorname{Per}_\Omega \{x : u(x) = 1\} & \text{if } u = \pm 1 \text{ ae} \\ \infty & \text{otherwise} \end{cases}$$

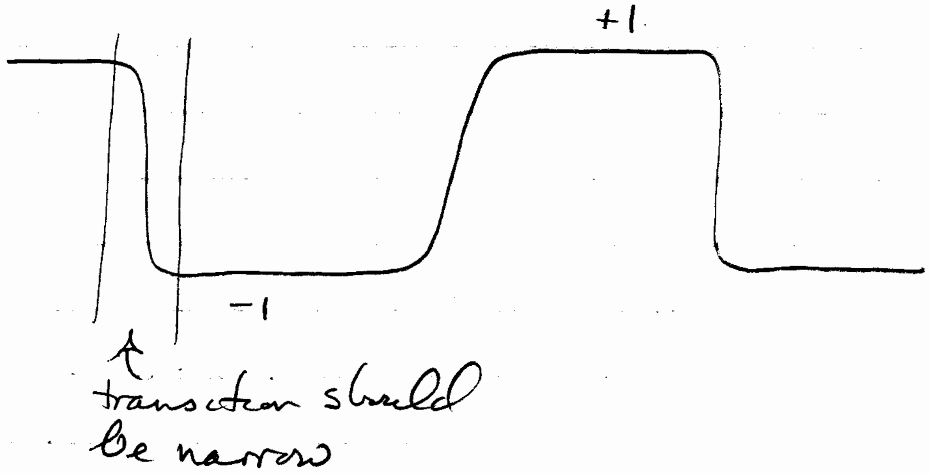
and as consequences:

a) if $G(u)$ is cpt under L^1 convergence, minimizers of $F_\varepsilon(u) + G(u)$ converge to minimizers of $F_0(u) + G(u)$

b) an isolated L^1-local min of $F_0(u)$ is the limit as $\varepsilon \to 0$ of L^1-local minimizers of F_ε.

(We'll explain these in due course; but note: the main point of Γ-convergence is to provide info about minimizers.)

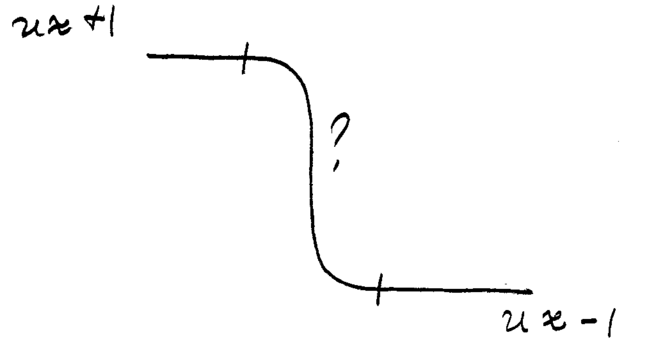
Some intuition first: suppose $\Omega = [0,1] \subset \mathbb{R}$ and ε is small. If F_ε is not of order $1/\varepsilon$ then $u^2 \approx 1$ so u should be close to ± 1 except on some "transitions"



1D Modica-Mortola asserts: as $\varepsilon \to 0$, energy required for a transition is exactly $8/3$.

It's easy to see why this should be true:

suppose $u \approx +1$ for $x \le a$ and $u \approx -1$ for $x \ge b$.



Then
$$\int_a^b \varepsilon u_x^2 + \frac{1}{\varepsilon}(u^2-1)^2 \ge 2\int_a^b |u^2-1|\,|u_x|$$
$$= 2\int_a^b |\varphi(u)_x| = 2|\varphi(1)-\varphi(-1)|$$

where $\varphi'(t) = |t^2-1|$, eg $\varphi(t) = \int_{-1}^{t} |1-s^2|\,ds$, for which

$$-1 \le t \le 1 \Rightarrow \varphi(t) = t - \tfrac{1}{3}t^3 \Big|_{-1}^{t} \Rightarrow \varphi(1)-\varphi(-1) = 4/3,$$

This calcn also shows form of the optimal transition: it has

$$\sqrt{\varepsilon}\,|u_x| = \frac{1}{\sqrt{\varepsilon}}|u^2-1| \qquad \text{ie} \qquad \varepsilon u_x = \pm(1-u^2)$$

This is easily integrated; one finds that

$$u \simeq \pm \tanh\left(\frac{x-x_0}{\varepsilon}\right)$$

for an (optimal) transition centered at x_0.

Preceding arg't was purely variational - there was no PDE in sight - but it is related to method of matched asymptotic expansions. Our 1D fnl

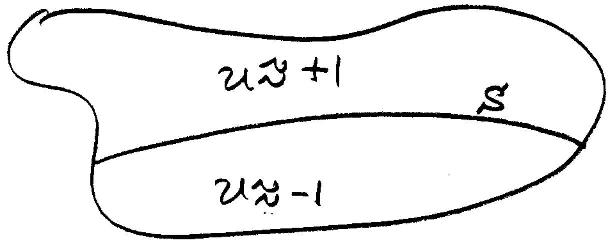
$$\int_0^1 \varepsilon u_x^2 + \tfrac{1}{\varepsilon}(u^2-1)^2$$

has only the obvious local minima $u \equiv \pm 1$ (this was an exercise earlier this semester) but it has plenty of saddle pts, which solve

$$-2\varepsilon u_{xx} + 4\varepsilon^{-1}(u^3-u) = 0 \quad \text{in } [0,1]$$
$$u_x = 0 \quad \text{at endpts}$$

Low-index saddles can be understood by phase plane analysis - the "transitions" are more or less evenly spaced. One could try to represent the soln of pde by matched asymptotic expansion. The "inner expansion" (describing a single transition btwn +1 & -1) would lead to same profile we derived above. (It's actually difficult to implement this fully, because the "outer expansion" involves only terms that are "exponentially small" as $\varepsilon \to 0$.)

Multidimensional picture is similar - let's focus on 2D for simplicity — but now "transitions" have more freedom (they're along curves, not at points). Still, assertion of Modica-Mortola is that to get



we require "energy" in the "transition layer near S" that's $8/3 \cdot \text{Length}(S)$.

Informal arg^t is as before:

$$\int_\Omega \varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}(u^2-1)^2 \geq 2\int_\Omega |\nabla u|\,|u^2-1|$$

$$= 2\int_\Omega |\varphi'(u)\nabla u|$$

$$= 2\int_\Omega |\nabla \varphi(u)|$$

If $u_\varepsilon \to u_0$ (discontinuous, as in picture, $u_0 = \pm 1$ a.e.) then $\varphi(u_\varepsilon) \to \varphi(u_0)$ (takes values $\varphi(1)$ + $\varphi(-1)$, jumping across S) and

$$2\int |\nabla\varphi(u_0)| = 2[\varphi(1)-\varphi(-1)]\cdot \text{Length}(S)$$

since $\nabla \Phi(u_0)$ is "a δ-fn concentrated at S."

Moreover, arg't shows that ineq is (almost) sharp if $u_\varepsilon(x) = \pm \tanh\left(d(x,S)/\varepsilon\right)$, where $d(x,S)$ = signed distance to S.

More careful statements:

① if $v_\varepsilon \to v_0$ in L^1 as $\varepsilon \to 0$, then

$$\liminf_{\varepsilon \to 0} F_\varepsilon(v_\varepsilon) \geq F_0(v_0)$$

② if $v_0 \in L^1$ then $\exists\, v_\varepsilon \to v_0$ in L^1 s.t.

$$F_\varepsilon(v_\varepsilon) \to F_0(v_0) \quad .$$

These constitute the <u>defn</u> of F_ε Γ-converging to F_0 in the L^1 topology. Also, we have

③ if $\{v_\varepsilon\}$ have $F_\varepsilon(v_\varepsilon)$ unif bdd, then $\{v_\varepsilon\}$ is cpt in $L^1(\Omega)$.

This is why it is natural, in this example, to study Γ-convergence wr to the L^1 topology.

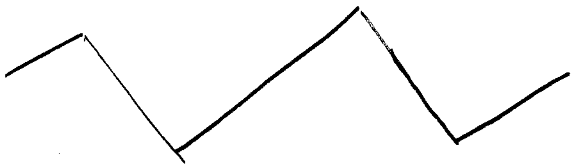
Our intuitive discn should have made ① – ③ plausible; we'll return to discuss them more carefully later. But now let's discuss their consequences

1st consequence (special case, in 1D, for simplicity): consider

$$E_\varepsilon(u) = \int_0^1 \underbrace{\varepsilon u_{xx}^2 + \frac{1}{\varepsilon}(u_x^2-1)^2}_{\substack{\text{Modica-Mortola} \\ \text{wrt } u_x\,!}} + \lambda u^2$$

In limit $\varepsilon \to 0$, it Γ-converges to

$$E_0(u) = \int_0^1 \lambda u^2 dx + \frac{8}{3}\cdot \#(\text{teeth}), \quad \text{if } u_x = \pm 1$$
(a "sawtooth").



In particular, $\min_u E_\varepsilon \to \min E_0$, and if u_ε minimizes E_ε then limit pts of $\{u_\varepsilon\}$ are minimizers of E_0.

(Note: for E_0, the opt'l # of teeth depends on

λ; finding the optimal u for a given # teeth is a nonlocal var'l pblm, since constraint $u_x = \pm 1$ is very rigid.)

Since it's u_x (not u) that participates in the "Modica-Mortola terms" $\varepsilon u_{xx}^2 + \frac{1}{\varepsilon}(u_x^2-1)^2$, the relevant topology is $u_x \in L^1$ ie $W^{1,1}$ (though the limits are better: $W^{1,\infty}$). The lower order term λu^2 is cpt in this topology, so it does not affect the Γ-convergence (if $F_\varepsilon \underset{\Gamma}{\Rightarrow} F_0$ as defined by ① + ② then $F_\varepsilon + G \underset{\Gamma}{\Rightarrow} F_0 + G$ provided G is cont's on $W^{1,1}$).

Claim: Γ-convergence $\Rightarrow$ convergence of minimizers, and convergence of minimizing value.

Pf: if $F_\varepsilon \underset{\Gamma}{\rightarrow} F_0$ then

$$\limsup_{\varepsilon \to 0} (\min F_\varepsilon) \le \min F_0 \qquad \text{by ②}$$

but

$$\liminf_{\varepsilon \to 0} (\min F_\varepsilon) \ge \min F_0 \qquad \text{by ①}$$

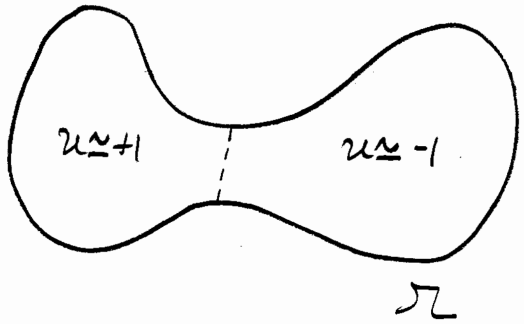
So minimizing values converge. So if u_ε minimizes F_ε, $F_\varepsilon(u_\varepsilon) \to \min F_0$. Now suppose $u_\varepsilon \to u_0$. Then

$$\liminf F_\varepsilon(u_\varepsilon) \ge F_0(u_0) \qquad \text{by ①}$$

Thus $F_0(u_0) = \min F_0$; so u_0 achieves $\min(F_0)$.

(<u>Note</u>: preceding pf is general, ie not special to Modica-Mortola.)

<u>2nd consequence</u> (drawn from the Kohn-Sternberg paper; see pg 12.3 for full citation). Suppose Ω has a "strictly convex neck"



Then

A) The fnl $F_0 = \frac{8}{3} \operatorname{Per}_\Omega \{u=1\}$ if $u = \pm 1$

has an isolated L^1-local min u_0 assoc to minimal-length path crossing the neck

B) $F_\varepsilon = \int_\Omega \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon}(u^2-1)^2$ has a local min u_ε st $u_\varepsilon \to u_0$ as $\varepsilon \to 0$.

Pf of (A) is not trivial, but I won't do it here (see the paper).

Pf of (B) is similar to what we did before: consider

$$\min_{\|u-u_0\|_{L^1} \le \delta} F_\varepsilon(u)$$

If ε is optimal, then either $\|u_\varepsilon - u_0\|_{L^1} = \delta$ or else it's a local min of F_ε in L^1 topology.

<u>Claim:</u> For small ε we cannot have $\|u_\varepsilon - u_0\|_{L^1} = \delta$.
<u>In fact</u>, if $\exists$ seq $\varepsilon_j \to 0$ st $\|u_{\varepsilon_j} - u_0\|_{L^1} = \delta$ then (using cptness) $\exists$ limit pt u_* st $\|u_* - u_0\|_{L^1} = \delta$, and part ① of Γ-conv defn says

$$\liminf_{\varepsilon_j \to 0} F_{\varepsilon_j}(u_{\varepsilon_j}) \ge F_0(u_*).$$

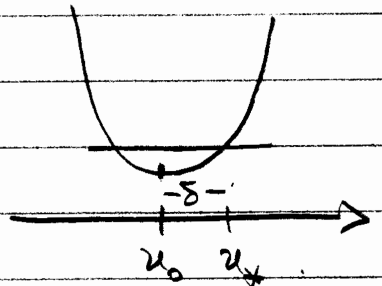
But pt ② of defn of Γ-conv tells us that

$$\limsup_{\varepsilon_j \to 0} F_{\varepsilon_j}(u_{\varepsilon_j}) \le F_0(u_0)$$

So

$$F_0(u_*) \le F_0(u_0).$$

But by (A), u_0 is an <u>L^1-isolated</u> local min of F_0, so if δ is small enough we get a contradiction (remembering that $\|u_* - u_0\|_{L^1} = \delta$).



schematic of F_0

(Claim is now proved.)

It's now clear that u_ε is an L^1-local min of F_ε for suff'tly small ε. To see that $u_\varepsilon \to u_0$, suppose it were not so. Then $\exists$ seq $\varepsilon_j \to 0$ s.t. $\|u_{\varepsilon_j} - u_0\|_{L^1} \geq c_0 > 0$. Then any limit pt u_* has

$\|u_* - u_0\|_{L^1} \geq c_0$. But, as shown above, we would have

$$F_0(u_*) \leq F_0(u_0)$$

for such a limit pt, contradicting again the hypoth that u_0 was an isolated local min.

It remains to explain how properties ①, ②, ③ (stated in pg 12.5) are proved for the Modica-Mortola functional.

The analysis uses some properties of the function space

$$BV(\Omega) = \left\{ u \in L^1(\Omega) \text{ st } \int_\Omega |\nabla u| < \infty \right\}$$

where we use the (slightly abusive) notation

$$\int_\Omega |\nabla u| = \sup_{\substack{g \in C_0^\infty(\Omega, \mathbb{R}^n) \\ |g| \le 1 \text{ ptwise}}} \int_\Omega u \operatorname{div} g \, dx$$

= total variation of the vector-valued measure "∇u"

Note that $|\nabla u|$ is not an L^1 fn, but rather a measure which acts on cont's fns by

$$\int_\Omega h |\nabla u| = \sup_{\substack{g \in C_0^\infty(\Omega, \mathbb{R}^n) \\ |g(x)| \le h(x) \text{ ptwise}}} \int_\Omega u \operatorname{div} g$$

By Green's formula, the char fn of a set A with smooth bdry is in BV, and

$$\operatorname{Per}_\Omega(A) = \int_\Omega |\nabla \chi_A| = \text{surface area of } (\partial A) \cap \Omega.$$

(Exercise: prove this, using Green's formula.)

Some (relatively easy) facts:

A) Lower semicontinuity: if $u_\varepsilon \to u$ in $L^1(\Omega)$, then

$$\liminf_{\varepsilon \to 0} \int_\Omega |\nabla u_\varepsilon| \geq \int_\Omega |\nabla u|.$$

This is obvious, since $\int |\nabla u|$ is a sup of cont's linear fnls on L^1.

B) Bounded sets in the BV norm ($\|u\|_{BV} = \|u\|_{L^1} + \int |\nabla u|$) are compact in L^1.

This is the BV analogue of the familiar stmt that the embedding $W^{1,p} \hookrightarrow L^q$ is compact for $\Omega \subset \mathbb{R}^n$ bounded and $q < \frac{np}{n-p}$. (Our case is $q=1$ and [roughly] $p=1$). But $BV \neq W^{1,1}$ so an honest pf must be different. Sketch of one argument:

step 1 show that $\inf_{\bar{u} \in \mathbb{R}} \int_Q |u - \bar{u}| \leq C \int_Q |\nabla u|$ when $Q = [0,1]^n$. This rescales to

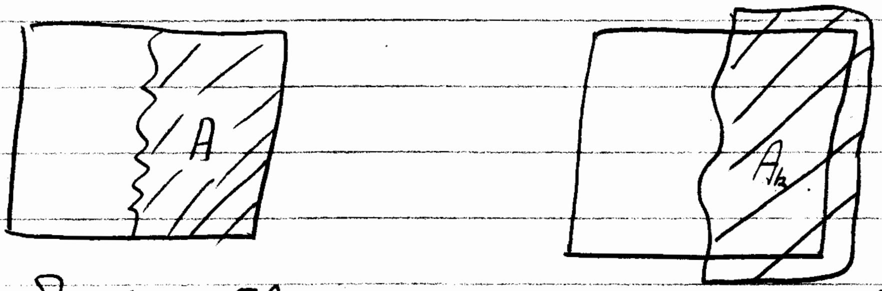
$$\inf_{\bar{u} \in \mathbb{R}} \int_{Q_\ell} |u - \bar{u}| \leq C \ell \int_{Q_\ell} |\nabla u|$$

when $Q = [0,\ell]^n$.

<u>step 2</u> It follows from step 1 that $u \in BV(\Omega) \Rightarrow u$ can be approx well in L^1 by a piecewise constant fn. (Obvious if Ω is a union of cubes; requires a bit of if not)

<u>step 3</u> The asserted compactness follows easily from this approxn lemma. (Exercise)

c) If a set A has bounded perimeter (ie its char fn $\chi_A \in BV$) then there are sets $A_k \subset \mathbb{R}^n$ with C^2 bdry st $\chi_{A_k} \to \chi_A$ in L^1 and $Per_\Omega(A_k) \to Per_\Omega(A)$ and $\mathcal{H}^{n-1}(\partial A_k \cap \partial\Omega) = 0$.



Ω = square
A = shaded

$\partial A_k \cap \Omega$ is C^2.
$Per_\Omega(A_k) \approx Per_\Omega(A)$.
∂A_k avoids $\partial\Omega$.

(note that taking $A_k \subset \Omega$ is <u>not</u> possible).

Proof of C uses mollification combined with the co-area formula.

One place to look for the above facts is the book of Jost + Li-Jost (it has complete discussions of facts (A) + (B) and all the ingredients needed to prove (C)).

Accepting (A) – (C) as above, let me now discuss the Γ-convergence (assertions ① + ②) + compactness (assertion ③) properties of the Modica-Mortola fnl.

Proof of ①: Observe first that it suffices to consider limits u st $u(x) = \pm 1$ ae, since if $u_\varepsilon \to u$ in L^1 then

$$\liminf_{\varepsilon \to 0} \int_\Omega \varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}(u_\varepsilon^2 - 1)^2$$

$$\geq \liminf_{\varepsilon} \frac{1}{\varepsilon} \int_\Omega (u_\varepsilon^2 - 1)^2$$

(Fatou's lemma)

$$\geq \int \liminf_{\varepsilon} \frac{1}{\varepsilon}(u_\varepsilon^2 - 1)^2$$

$= \infty$ if u takes values other than ± 1 on a set of pos measure

Next: observe it suffices to consider u_ε st

$$-1 \leq u_\varepsilon \leq +1$$

since otherwise we can replace u_ε by its truncation

$$u_\varepsilon^* = \begin{cases} +1 & u_\varepsilon(x) > 1. \\ u_\varepsilon(x) & -1 \leq u_\varepsilon(x) \leq +1. \\ -1 & u_\varepsilon(x) < -1. \end{cases}$$

without changing the L^1 limit (this truncation decreases the value of F_ε).

Finally, suppose $-1 \leq u_\varepsilon(x) \leq 1$ ae and $u_\varepsilon \to u_0$ in L^1, with $u_0 = \pm 1$ ae. Then

$$F_\varepsilon(u_\varepsilon) \geq 2 \int_\Omega |\nabla \varphi(u_\varepsilon)|.$$

where $\varphi(t) = \int_{-1}^{t} |1-t^2|\, dt$. (we explained this earlier). By dominated convergence, $\varphi(u_\varepsilon) \to \varphi(u_0)$ in L^1. By lsc of BV norm,

$$\liminf F_\varepsilon(u_\varepsilon) \geq 2 \int_\Omega |\nabla \varphi(u_0)|$$

But

$$2\varphi(u_0) = \begin{cases} 0 & \text{where } u_0 = -1. \\ 8/3 & \text{where } u_0 = +1 \end{cases}$$

so

$$2\int_\Omega |\nabla\varphi(u_0)| = \frac{8}{3}\, \mathrm{Per}_\Omega\{u_0=1\}.$$

<u>Sketch of pf of (2)</u>: As explained earlier, for equality to hold in

$$\int \varepsilon|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}|u_\varepsilon^2-1|^2 \geq 2\int|\nabla\varphi(u_\varepsilon)|$$

we would need $\varepsilon|\nabla u_\varepsilon| \simeq |u_\varepsilon^2-1|$. In 1D this is achieved by using $u_\varepsilon \simeq \pm\tanh\left(\frac{x-x_j}{\varepsilon}\right)$ near the j^{th} transition (modified far from x_j so that $u_\varepsilon = \pm 1$ exactly). In $\mathbb{R}^n$, $n\geq 2$, one gets something similar (as $\varepsilon\to 0$, with the geometry smooth enough + held fixed) by using

$$u_\varepsilon \simeq \tanh\left(\frac{\mathrm{dist}(x,A_k)}{\varepsilon}\right).$$

Since by fact (C) above it suffices to consider smooth sets A_k, this works. (Making this argument fully precise is a bit tedious.)

<u>Proof of (3)</u>, ie compactness in L^1: recall that

we have a unif bd on $\int_{\Omega} |\nabla \varphi(u_\varepsilon)|$ where φ is monotone, $\varphi'(t) = |t^2-1|$.

Using the form of φ, we have $\varphi(t) \leq C(1+|t|^3)$ so

$$\int_{\Omega} \varphi(u_\varepsilon) \leq C \int_{\Omega} 1 + |u_\varepsilon|^3$$

$$\leq \text{const indep of } \varepsilon.$$

using that $\frac{1}{\varepsilon}\int_{\Omega}(u_\varepsilon^2-1)^2$ stays bounded, and $|u^2-1|^2 \sim |u|^4$ if $|u| \gg 1$.

So $\{\varphi(u_\varepsilon)\}$ stays bdd in BV, whence $\{\varphi(u_\varepsilon)\}$ is precompact in L^1: any sequence has a subsequence st

$$v_{\varepsilon_j} = \varphi(u_{\varepsilon_j}) \to v_0 \quad \text{in } L^1$$

By unif cont'y of φ^{-1} we get

u_{ε_j} converges in measure to $\varphi^{-1}(v_0)$.

Since u_{ε_j} are unif bdd in L^4, it follows that they converge to $u_0 = \varphi^{-1}(v_0)$ in L^1.
(Details: write

$$\int_\Omega |u_{\varepsilon_j} - u_0| = \text{integral over } \Omega \cap \{|u_{\varepsilon_j} - u_0| \le \delta\}.$$
$$+ \text{integral over } \Omega \cap \{|u_{\varepsilon_j} - u_0| \ge M\}.$$
$$+ \text{integral over } \Omega \cap \{\delta < |u_{\varepsilon_j} - u_0| < M\}.$$

1^{st} term $\le \delta \cdot vol(\Omega)$.

2^{nd} term $\le C M^{-3}$ since $|u_{\varepsilon_j} - u_0| \le M^{-3} |u_{\varepsilon_j} - u_0|^4$ on this set and we have a unif bd in L^4.

3^{rd} term $\to 0$ as $\varepsilon_j \to 0$ by conv. in meas, [or by conv of $\varphi(u_{\varepsilon_j})$ to $\varphi(u_0)$ in L^1].

Evidently $\limsup_{j\to\infty} \int |u_{\varepsilon_j} - u_0| \le C(\delta + M^{-3})$.
Letting $\delta \to 0$ and $M \to \infty$ gives the result.)

See next pg for suggested exercises (note: They intentionally focus on the 1^{st} part of this lecture, not the pde analysis-heavy last part.)

(1) Recall our discussion why, in the 1D setting, if $u(a)=1$ and $u(b)=-1$ then

$$\int_a^b \varepsilon u_x^2 + \frac{1}{\varepsilon}(u^2-1)^2 \geq \frac{8}{3} ,$$

the inequality being very nearly sharp when u has a tanh profile (adjusted slightly near a + b to meet the bc)

Show that something similar can be done for

$$\int_a^b \varepsilon u_x^2 + \frac{1}{\varepsilon} W(u)\, dx$$

when W is smooth with

$$W \geq 0, \text{ and } W = 0 \text{ exactly at } u = \pm 1 .$$

(Note, however, that it makes a difference whether $W''(\pm 1)$ is strictly positive or not, since the way that $\int_0^a \frac{du}{W^{1/2}(u)} \to \infty$ as $a \to 1$ depends on this.)

(2) Now consider the analogue of pbm 1 where u is vector-valued, $u: [a,b] \to \mathbb{R}^n$, and the "potential" $W: \mathbb{R}^n \to \mathbb{R}$ prefers two points

in $\mathbb{R}^n$

$W \geq 0$, with $W = 0$ exactly at $\vec{u} = \xi$ + $u = \eta$

Show that

$$\min_{\substack{u(a)=\xi \\ u(b)=\eta}} \int_a^b \varepsilon u_x^2 + \frac{1}{\varepsilon} W(u) \geq d$$

where

d = "distance from ξ to η, in the metric on $\mathbb{R}^n$ with weight $W^{1/2}$"

$$= \min_{\substack{\vec{y}(0)=\xi \\ \vec{y}(1)=\eta}} \int_0^1 W^{1/2}(\vec{y}(t)) \, |\dot{y}(t)| \, dt.$$

(Hint: show that $W^{1/2}(u)|u_x| \geq |\partial_x \varphi(u)|$ for any $\vec{u} \in \mathbb{R}^n$, when

$\varphi(\vec{u})$ = "distance from ξ to $\vec{u}$, in the metric on $\mathbb{R}^n$ with weight $W^{1/2}$"

$$= \min_{\substack{\vec{y}(0)=\xi \\ \vec{y}(1)=\vec{u}}} \int_0^1 W^{1/2}(\vec{y}(t)) \, |\dot{y}(t)| \, dt \quad .)$$