Calculus of Variations, Spring 2013
MATH-GA 2660.001 (Advanced Topics in Analysis)
Bob Kohn, kohn@cims.nyu.edu
Tuesdays 1:25-3:15, WWH 512
End-of-semester version, reflects actual coverage

Prerequisites: Real Variables I and PDE I (or equivalent).

Calendar: First class 1/28/2013. No class 3/19 (spring break). Last class 5/7.

Course Description: A modern introduction to the Calculus of Variations, with equal emphasis on theory and applications. Topics will include: existence of solutions and convergence of numerical schemes; convex duality; one-dimensional variational problems; multidimensional nonconvex problems; relaxation; Gamma convergence; and length scale selection via singular perturbation. Along the way, we'll discuss many applications including minimal surfaces, optimal control, nonlinear elasticity, optimal design, martensitic phase transformations, and the wrinkling of thin sheets.

All lectures will be supported by (handwritten) lecture notes, which will include selected exercises (recommended but not required) and suggestions where to read more. There will be no final exam.

There will be a little overlap with the material covered in Mechanics – mainly concerning 1D variational problems (which arise in Hamiltonian mechanics through action minimization) and elasticity (which is, for us, a source of examples). The overlap will be limited – about 2 lectures – so the two classes are more complementary than redundant.

Website: Lecture notes will be posted on my website (see math.nyu.edu/faculty/kohn/teaching.html). I'll also use an NYU-Classes site, mainly for communication by email, and for posting material meant only for students in the class. Non-registered auditors can ask to be added to the NYU-Classes site.

Office hours: My office hours are Tues & Wed 11-12 (WWH 502).

Semester plan:

- The direct method of the calculus of variations (1/29)
- Convex duality (2/5, 2/12)
- One-dimensional variational problems (2/19, 2/26)
- Optimal control: Hamilton-Jacobi equations and Pontryagin's Maximum Principle (3/5, 3/12, 3/26)
- Some key multidimensional examples: elasticity, thin sheets, optimal design, phase transformation (3/26, 4/2)

- Relaxation and quasiconvexity (4/9, 4/16)
- Intro to Gamma-convergence, via (i) the Modica-Mortola problem, and (ii) thin elastic structures (4/23, 4/30)
- Singular perturbation and the length scale of microstructure (5/7)

This outline omits many important themes in the Calculus of Variations, including the mountain pass lemma, regularity of minimizers or critical points, harmonic maps, homogenization, and Ginzburg-Landau vortices. One semester is short!

Library reserve: The following books are being placed on reserve in the Courant library:

- G. Buttazzo, M. Giaquinta, and S. Hildebrandt, *One dimensional variational problems: an introduction*, Oxford University Press 1998
- B. Dacorogna, *Direct methods in the calculus of variations*, Springer-Verlag, 2nd edition, 2008 – also available as an ebook via Bobcat
- A. Dixit, *Optimization in economic theory*, Oxford Univ Press 1990
- I. Ekeland and R. Temam, *Convex analysis and variational problems*, North Holland, 1976
- R. V. Gamkrelidze, *Principles of optimal control theory*, Plenum Press, 1978
- L. Hocking, *Optimal control: An introduction to the theory, with applications*, Oxford Univ Press 1991
- J. Jost and X Li-Jost, *Calculus of variations*, Cambridge University Press
- M. Mesterton-Gibbons, *A primer on the calculus of variations and optimal control theory*, Amer Math Soc 2009

The following book is not on reserve, because you have access to the ebook version (via Bobcat):

- A. Braides, *Gamma-convergence for beginners*, Oxford University Press 2002

For some topics, you might find Evans' PDE book a convenient source:

- L.C. Evans, *Partial differential equations*, Amer Math Soc, 2nd edn, 2010

Buttazzo-Giaquinta-Hildebrandt and Mesterton-Gibbons provide complementary introductions to the 1D calculus of variations (Mesterton-Gibbons is more elementary and example-oriented); this topic is also covered well in Chapters 1 & 2 of Jost and Li-Jost. Our discussion of convex duality will be much more concrete and basic than that of Ekeland-Temam, but students who want to know more may find that source useful. On optimal control, Evans is best for the role of Hamilton-Jacobi equations, while Dixit and Hocking focus on the Pontryagin maximum principle (with lots of examples). Gamkrelidze discusses the existence of optimal controls, using a viewpoint similar to "relaxation," but we won't have time for such advanced material. Dacorogna provides a good treatment of lower semicontinuity and relaxation in the multidimensional setting. Braides gives a good introduction to the main ideas of Gamma-convergence, including the Modica-Mortola problem, but is restricted to the 1D setting. Jost and Li-Jost touches many of our topics, but (except for the initial chapters and the discussion of Modica-Mortola) I don't find it very readable. Some of our examples (thin sheets, optimal design, phase transformation, singular perturbation and the length scale of microstructure) will draw more on articles than on the books listed above.