in: Sleeman + Jarvis eds, Theory of Ordinary + Partial Differential Eqns, Longman, 1986

FIBERED STRUCTURES IN OPTIMAL DESIGN

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Our starting point is the following question:
Given a differential equation div $\sigma = 0$, an inhomogeneous boundary condition $\sigma \cdot n = f$, and an upper bound $|\sigma| \leq c$, find the vector field σ with the smallest possible support.
In mechanical terms, this asks:
Among all stresses that are in equilibrium with a given surface force f and subject to a plastic yield limit $|\sigma| \leq c$, find the stress σ which vanishes in the largest possible set.
Or, more briefly, since we can remove any part of a given plastic structure in which the stresses are zero:
Find the lightest substructure which will withstand the given load.

This is one form of the fundamental problem of optimal design. It differs from the usual problem of stress analysis in which loads, the shape of the structure, and the properties of the material are all prescribed. In that case the stress field is the solution to an elliptic boundary value problem. In our case only the load is prescribed, together with constraints on the geometry and the material. The shape is then chosen to minimize the weight. We shall see that in the extreme case when the design is optimal, its shape can become non-standard and so can the material.

This paper begins with one case of the design problem: an infinite plastic cylinder subject to antiplane shear. That problem has the great advantage that the unknown stress has only two nonzero components $\sigma_1 = \sigma_{xz}(x,y)$ and $\sigma_2 = \sigma_{yz}(x,y)$. The displacement is a scalar $u(x,y)$, and so is the stress function $\psi(x,y)$. We studied this problem earlier, and we return to it here because it brings out most clearly the fundamental steps in the analysis:

1. The weight minimization problem is not convex.

2. Its relaxation to a convex problem (which has the same minimum) is equivalent to the introduction of fibered composites. Instead of a 0-1 choice between no material and the original material, there is at each point a continuum of choices: the density and direction of the fibers are to be optimized.

3. This relaxed (or convexified) problem is well-posed. The admissible ψ are functions of bounded variation over Ω. The coarea formula leads to the solution--in examples as well as theory--and also to its regularity: stress trajectories are slightly better than C^1.

It will become evident that the optimal designs are extremely difficult, in other words impossible, to manufacture. The closest approach may be MacReady's Gossamer Albatross, the 55-pound aircraft which crossed the English Channel powered by a cyclist. It was designed by iteration, testing each model in flight and strengthening those parts that were responsible for a crash. (The parts that never broke were assumed to be too heavy, and were weakened.) More recently there are jet planes in which substantial sections are made of fibered material, to add strength without weight. Composites are increasingly important for automobiles. Fortunately the optimum is sufficiently flat that a rough approximation to the best design (which may require curved fibers) is not much heavier than necessary.

The relaxation of this problem coincides with the convexification of the cost functional, since the unknown ψ is a scalar. For a vector unknown (a stress tensor or a displacement in several dimensions) that is no longer true. If $\iint F(u_x,u_y,v_x,v_y)\,dxdy$ is nonconvex, we cannot replace F by the largest convex function below it. In the scalar case $\iint F(\psi_x,\psi_y)\,dxdy$, we will see how oscillations in ψ make the replacement correct; the infimum is not changed. In the vector case those oscillations cannot achieve so great a cost reduction, and the relaxed form becomes the quasiconvexification of the original problem.

That vector case has been the focus of our recent work [1]. It can be approached as an application of the Tartar-Murat theory of compensated compactness [2-3], whose goal is exactly what was required above: to determine what equations and inequalities remain correct in the weak limit, when oscillations increase in frequency and only their averages are computable. One such average leads to the effective properties of a homogenized composite--the limiting case of oscillations in the material. The optimal composites achieve the bounds on effective properties when the fractions of each material are prescribed [4-6]. (We have also obtained those bounds independently of the compensated compactness technique, by verifying that the quasiconvexification had been reached [7].) Then the optimal design is found by choosing at each point from the collection of optimal composites. The local constructions are combined into a global construction which withstands the imposed load and has minimum weight.

In this paper we stay with the scalar problem, and give the construction that justifies convexification. It is comparable to the "truss continuum" which appears first in the remarkable work of Michell [8]. He discovered that the bars in an optimal truss form a special orthogonal net, typified by pairs of spirals or by tangents and involutes. For the geometry we refer to Hill, Prager, Geiringer, and Kachanov. Then Prager led the subject of optimal design into a much wider class of applications, extending a sufficient condition discovered by Mroz to test the optimality of a proposed shape. Their criterion requires a function with $|\text{grad } u|$ = constant on the boundary; we have found that function in analyzing the dual problem. In general its gradient is constant over a region of positive measure--exactly the region where the fiber density is between zero and one. This generalizes the sharp line that Prager had hoped might mark out the optimal design. The same possibility was recognized in plate problems, where numerical computations indicated an infinite sequence of infinitely thin stiffeners. A survey of engineering applications was given by Rozvany [9], and Dacorogna provided an exposition of the theory of weak lower semicontinuity [10]. We hope to contribute one further note on the plate problem.

There is also another family of applications, more exciting but much more uncertain. Those are the designs that occur in biology. It is natural to hope that our hearts are optimally designed, and our bones, although the criteria for optimality must be much more complicated than minimum weight. In fact it is as much the problems as the solutions that are unknown: What quantity is minimized, and what are the constraints? In the heart muscle we find fibers that are wound in a definite pattern, and in bones it has been observed for 100 years that the trabecular fibers in the spongy part seem to be aligned with the directions of maximum stress. This is known to orthopaedic surgeons as Wolff's law. The growth or decay of bones depends directly on the external loads, and there is a major effort to make this relation of stress to morphology quantitative rather than qualitative [11-12]. It seems likely that optimal designs which are virtually impossible to manufacture have nevertheless been evolved.

The Design of the Butterfly

We return to the model problem of antiplane shear. The external shearing force is $\sigma \cdot n = f$, on the boundary of an infinite cylinder of cross-section Ω. All functions are independent of the axial variable z, and the stress is in equilibrium:

$$\operatorname{div} \sigma = \frac{\partial \sigma_1}{\partial x} + \frac{\partial \sigma_2}{\partial y} = 0 \quad \text{in the cylinder.} \tag{1}$$

The von Mises yield condition can be normalized to

$$|\sigma| = (\sigma_1^2 + \sigma_2^2)^{1/2} \leq 1. \tag{2}$$

Our problem is to find the largest subset of Ω in which σ can be zero. In that set no material is required.

We take Ω to have the shape of a butterfly. The surface forces are zero on the sloping boundaries and $f = \pm w/\ell$ on the wing tips. The net force on the boundary is zero, in agreement with $\iint \operatorname{div} \sigma \, dxdy = 0$ in the interior. Provided the sine of the corner angles is not less than w/ℓ, there exists a stress field σ that satisfies the requirements (1) and (2). (The existence is a problem in plastic limit analysis [13].) In a region where the stress is below the limit, it can be modified by any divergence-free field. It is natural to suppose that in the optimal design, the stress is either at its maximum (where $|\sigma| = 1$) or its minimum (where $\sigma = 0$ and the material is replaced by a hole). That expectation is verified below, but only in the limit of infinitely many holes. There are examples in which a single hole is optimal, and $|\sigma|$ drops abruptly from 1 to 0, but they are rare.

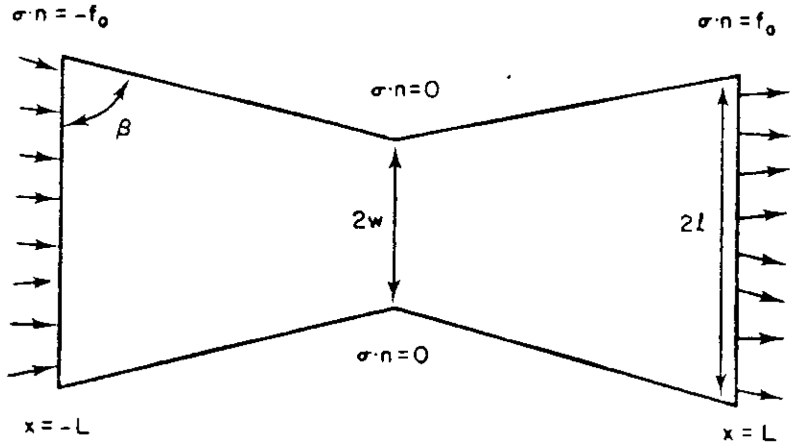

FIG. 1. The butterfly.

There is a natural analogy with the flow of traffic. The continuity condition $\operatorname{div} \sigma = 0$ means that no cars disappear. The constraint $|\sigma| \le 1$ limits the capacity. Across the centerline the flow cannot exceed $2w$, and that is exactly the flux $\int \sigma \cdot n = 2\ell(w/\ell)$ through the edges. Therefore we must have the maximum possible stress $\sigma = (1,0)$ along the centerline. This is completely analogous to the max flow-min cut theorem of network flow, and a continuous version of that theorem [14] shows that the required flow w/ℓ can be achieved with $|\sigma| \le 1$.

We can remove part of Ω only if no bottleneck is created. Figure 2 illustrates one possibility--to channel the flow along the boundary, leaving a wedge-shaped region of "stagnation" in the center of each wing. If the remaining strips have width w then the flow is still feasible; in terms of stresses, there is still an admissible σ. Note that we take $\sigma \cdot n = 0$ on the boundary of such a hole; no flow crosses it in the traffic analogy, and it supports no shear stress. Near the center there are circular sectors of radius w. almost indistinguishable from triangles in the figure, where the flow can change direction; it passes horizontally through the centerline and turns until it is parallel to the sloping boundary. The real difficulty is at the far right, where the flow needs enough space to achieve a uniform value of $\sigma \cdot n$ along the wing tip.

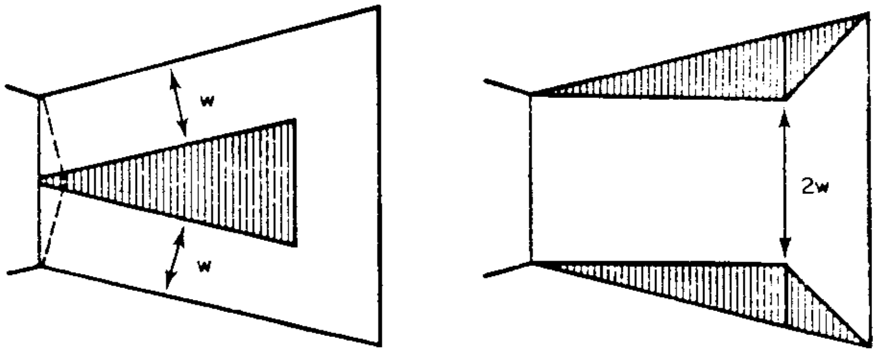

FIG.2a. Wedge-shaped hole. FIG.2b. Peripheral holes.

A second possibility is to remove material along the boundary and direct the traffic through the center. In this case (Figure 2) the holes touch the boundary, which is permitted where $f = 0$. Again it is at the far right, where the flow must reorganize itself to exit correctly, that we have kept more space than necessary. Roughly speaking, we can continue to introduce holes as long as any arc from the upper to the lower boundary is left with length $2w$, and any arc from the top or bottom to the vertical side remains long enough to admit the traffic that must get through. On the other hand, a design might achieve these minimum widths and not the minimum area. We could start from 2a or 2b and reach a point where no more holes are possible--but neither of the final designs would be optimal.

To make a better start, we can introduce holes that are more nearly aligned with the direction of flow. That design needs less space for adjustments at the boundary; the flow is close to the uniform distribution required of $\sigma \cdot n$. The more holes we use, the closer we come to an optimal flow. In fact, we are approaching a design which is entirely different from our initial experiments. In the limit, there is a density of material which varies from point to point and reaches its maximum (unity) near the center of Ω. Effectively, we have created a new material with its own capacity c. More than that, the final result is totally anisotropic. It is precisely analogous to a composite material produced by laying down a family of one-dimensional fibers. They have tremendous strength, but only in the direction of the fibers.

This conclusion, that there exists no optimal shape of an ordinary kind, will not surprise a specialist in the homogenization of domains. There it is the periodic case which is best understood; holes of fixed shape, and of size $1/n$, produce in the (weak) limit a material with properties entirely different from the original. In our case the holes are increasingly thin and the domain becomes foliated, reducing the strength to the minimum required.

The Solution in a Square

The first step toward the optimal design is the solution of a local problem--to replace a uniform stress that is below the limit $|\sigma| = 1$ by a stress distribution that is zero as often as possible. The region is a small square, but we may take it to be the square $-1 < x,y < 1$. The forces on the left and right sides are $\sigma \cdot n = \pm f_0$. The simplest solution is $\sigma = (f_0, 0)$, a uniform tension. It is typical of elastic solutions; it minimizes $\iint |\sigma|^2 \, dxdy$, it is the gradient of a harmonic function, but its support is the whole square. Therefore we look for other solutions.

In principle the total force $2f_0$ on the sides could be balanced by $\sigma = (1,0)$ in a band of width $2f_0$ through the center of the square. In this case σ is nonzero over an area of $4f_0$. That is the minimum possible. But this stress distribution is entirely wrong at the boundary--it has $\sigma . n$ equal to ± 1 at the band and zero outside it. We need a more careful construction if $\sigma \cdot n$ is to be uniform.

The optimum is approached by more and thinner bands. We use N bands of height $2f_0/(N-1)$, distributed evenly over the square. The stress inside them is $\sigma = ((N-1)/N, 0)$. Each band will end at some distance d from the edges of the square, and the real problem is to modify σ in that "boundary layer." Actually that problem on a small scale is identical to the butterfly problem on a larger scale--to convert a high uniform stress at the centerline to a lower uniform stress at the longer outside edge. The place of w is taken by $f_0/(N-1)$, the place of ℓ is taken by $1/N$, and the width L in

Fig. 1 is now d.

One solution is to carry the force on straight but sloping fibers. The resulting stress distribution is

$$\sigma = \frac{N-1}{N} \left(\frac{1}{1+\sqrt{N}\, x} , \frac{\sqrt{N}\, y}{(1+\sqrt{N}\, x)^2} \right) . \tag{3}$$

The denominator is smallest at $x = 0$ and the numerator is largest at $y = 1/N$, so that

$$|\sigma|^2 \leq \left(\frac{N-1}{N}\right)^2 \left(1 + \frac{1}{N}\right) < 1 .$$

The width d is determined by the boundary condition $\sigma \cdot n = f_0$ at the right side:

$$\frac{N-1}{N} \frac{1}{1+\sqrt{N}\, d} = f_0 \quad \text{or} \quad d = \frac{1}{\sqrt{N}} \left(\frac{N-1}{Nf_0} - 1\right) \leq \frac{\text{constant}}{\sqrt{N}} . \tag{4}$$

This stress field σ is also consistent with the uniform stress in the band at the left side. (Across a line of discontinuity it is only the normal stress $\sigma . n$ which must agree, and $(N-1)/N$ is the correct value.) The divergence of σ is seen to be zero, either by direct calculation or by recognizing that

$$\sigma = (\psi_y, -\psi_x) \quad \text{for} \quad \psi = \frac{N-1}{N} \frac{y}{1+\sqrt{N}\, x} .$$

This "stress function" is analogous to the "stream function" for fluids. It is constant on the lines of stress, which cross the layer of width d. The upper line goes from $x = 0$, $y = f_0/N-1$ (the top of the band) to $x = d$, $y = 1/N$ (at the outer boundary). Thus the proposed stress field, partly concentrated in N bands, partly distributed in the boundary layer according to (3), and partly zero, is admissible.

The area of N strips of height $2f_0/(N-1)$, together with the thin regions at the side, is not more than

$$2N \frac{2f_0}{N-1} + 4d \to 4f_0 .$$

In the limit, the cross-section therefore achieves the same area as the earlier attempt with $\sigma = (1,0)$ --while keeping $\sigma \cdot n$ equal to f_0 at the boundary. In fact the weak limit of the sequence σ_N is the elastic solution $(f_0, 0)$, but that happens only in special cases. What is more typical, and more important for the general case, is that the fraction of area required is given by f_0. Roughly speaking, a uniform stress of magnitude $|\sigma|$ can be replaced by stresses of unit magnitude concentrated over a fraction $|\sigma|$ of the original area.

To reach a variational formulation of our original problem, we extend this principle to an arbitrary domain Ω. Suppose there exists a stress σ that satisfies the three conditions $\text{div}\ \sigma = 0$, $\sigma \cdot n = f$, and $|\sigma| \leq 1$. Locally, in a small square about the point (x,y), we replace the given σ by stresses like σ_N --while retaining $\sigma_N \cdot n = \sigma \cdot n$ on the boundary of the square. The rest of Ω notices no change. The new σ_N is nonzero only in a fraction approaching $|\sigma(x,y)|$ of the square around (x,y), and the same construction takes place throughout Ω. Therefore the given σ can be replaced by stresses which require, in the limit, only the area $\iint |\sigma(x,y)|\, dxdy$. This is the relaxed functional mentioned in the introduction.

The Relaxed Problem

The minimum area of the support of σ was originally described as

$$\inf \iint 1_{\{\sigma \neq 0\}}\, dxdy, \text{ subject to } \sigma \cdot n = f,\ \text{div}\ \sigma = 0,\ |\sigma| \leq 1.$$

In this form the integrand is 1 whenever the stress is nonzero and material is needed. The magnitude of the stress is irrelevant (as long as it does not exceed the yield stress: $|\sigma| \leq 1$). In the relaxed problem the integrand is reduced to the magnitude $|\sigma|$:

$$\inf \iint |\sigma(x,y)|\, dxdy, \text{ subject to } \sigma \cdot n = f,\ \text{div}\ \sigma = 0,\ |\sigma| \leq 1.$$

Note that the solution σ^* to this relaxed problem is not optimal for the original problem. The importance of σ^* is to indicate a minimizing sequence for the original problem. The distinction comes in regions where $0 < |\sigma^*| < 1$, and the stress σ^* is replaced locally by the construction that was carried out above--fibers which lie in the direction of σ^* and have density $|\sigma^*|$. No replacement is needed in regions where $|\sigma^*| = 1$ or $\sigma = 0$; those are full of material, or empty. As $N \to \infty$ in the construction, the fiber area approaches the minimum that can be achieved and the stress field (which oscillates between $|\sigma| = 1$ in the fibers and $\sigma = 0$ in the holes) approaches a weak limit. That minimum area is $\iint |\sigma^*|\, dxdy$ and that weak limit is σ^*.

We want to recognize the relaxed problem as the convexification of the original. That is easy to do. The original integrand was 0 when $\sigma = 0$, 1 when $0 < |\sigma| \leq 1$, and $+\infty$ for $|\sigma| > 1$. It was nonconvex because of the jump from 0 to 1. The largest convex function below it has that jump removed; it is equal to $|\sigma|$ for $|\sigma| \leq 1$. The graph goes linearly between the points (0,0) and (1,1). (It can still jump to $+\infty$ for $|\sigma| > 1$ and remain convex.) The effect of our construction was to convexify the problem. That was established for a wide class of nonconvex variational problems by Ekeland-Temam [14] in the scalar case and by the present authors [1] in the vector case.

It remains to solve the relaxed problem. Our earlier paper [13] linked σ to the scalar stress function ϕ, and outlined the solution. A forthcoming article [15] will go farther. Here we give the main idea.

Each divergence-free σ is connected to a stress function ϕ by $\sigma = (\phi_y, -\phi_x)$. The divergence is zero because $\phi_{yx} = \phi_{xy}$. The boundary condition $\sigma \cdot n = f$ is changed to $\phi = g$, the indefinite integral of f along the boundary. (It is single-valued because $\oint f ds = 0$.) The magnitude $|\sigma|$ is $|\nabla\phi|$, and the problem becomes

$$\min \iint |\nabla\phi|\, dxdy \quad \text{subject to} \quad \phi = g \ \text{on} \ \partial\Omega \ \text{and} \ |\nabla\phi| \leq 1 .$$

This is the least gradient problem, constrained by $|\nabla\phi| \leq 1$. The existence of an admissible ϕ --so that $\phi = g$ is compatible with $|\nabla\phi| \leq 1$ --is the limit analysis problem that is solved by a continuous max flow-min cut theorem. The construction of the minimizing ϕ^* --whose gradient, rotated through $\pi/2$, is σ^* --can be carried out explicitly in many examples.

In fact our construction in the "small butterfly problem," connecting the uniform stress $(N-1)/N$ in the fiber to uniform stress along the boundary, was optimal for that problem. It came from a stress function ψ that was constant on straight lines in the region where $|\nabla\psi| < 1$ --which in that problem was everywhere.

In the original butterfly problem, that solution is not optimal. It is not even admissible. To achieve the boundary values $\sigma \cdot n = \pm\, w/\ell$, which is at the extreme limit, we had to maintain the maximum stress $|\sigma| = 1$ along the centerline. There was no factor $(N-1)/N$ to keep $|\nabla\psi|$ below one. There is a region near the center of the optimal design (the solid region in Fig. 3) which is filled with material. We can think of fibers of unit density, but they are curved. In the main part of the wing they are straight, and the stress decreases along the fiber until it satisfies $\sigma^* \cdot n = w/\ell$ at the wing tip. There is no region in the butterfly with $\sigma^* = 0$.

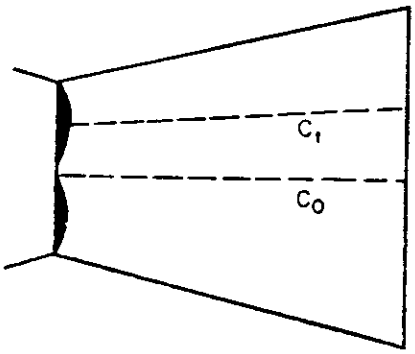

FIG.3. Fibers in the optimal design.

This description can be made more precise. The boundary values of ψ are $g = yw/\ell$ (the integral of $\sigma \cdot n$) on the wing tips, and $g = \pm\, w$ on the sloping edges of the butterfly. Suppose the fiber that meets the boundary point (L,y) makes an angle θ with the horizontal. At the boundary, $|\sigma| \cos\theta = w/\ell$. At the solid center, $|\sigma| = 1$. Where the line meets the solid center, it is tangent to a circular arc of radius $w - yw/\ell$ around the boundary point on the centerline. That determines the angle θ, and the fiber then curves to follow the circular arc. On the straight fiber, the reciprocal $1/|\sigma^*| = 1/|\nabla\psi^*|$ is a linear function of distance along the line. Therefore we can finally determine σ^*.

More important than these calculations is the form of the optimal design. It looks strangely like a butterfly, for no clear reason. The shear force on an infinite cylinder is not the one supported by a real butterfly. But it does have the attractive geometrical features--in this case a net of fibers and their orthogonals, in which the fibers are straight lines in one region and the orthogonals are straight lines in the other--which we are happy to find in nature.

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