Calculus of Variations, Lecture 2+3, 2/5/2013 + 2/12/2013

[Start by discussing material at end of "Lecture 1" notes, on "numerical perspective," pp 1.19 - 1.21]

After-thoughts to our discussion of the Direct Method:

(1) We know from undergrad Calculus that constraints typically lead to Lagrange multipliers. Why is there no "Lagr mult" assoc with a Dirichlet bc?

Short answer: when we solve

$$\min_{u=g \text{ at } \partial\Omega} \int_\Omega \tfrac{1}{2}|\nabla u|^2 + f u$$

one can view the normal deriv $\partial u/\partial \nu \big|_{\partial\Omega}$ as something like a Lagr mult assoc to the Dirichlet bc

Explanation: in $\mathbb{R}^n$, when we solve

$$\min_x F(x) + \frac{1}{\varepsilon} g^2(x)$$

we expect as $\varepsilon \to 0$ to get approx the soln of the constrained pblm

$$\min_{g=0} F(x)$$

Opt cond for former is $\nabla F + \frac{2}{\varepsilon} g \nabla g = 0$.

Opt cond for latter is $\nabla F + \lambda \nabla g = 0$.

Evidently, as $\varepsilon \to 0$ we expect $g(x_\varepsilon) \to 0$ (x_ε = opt'l choice of x) and

$\frac{2}{\varepsilon} g(x_\varepsilon) \to$ Lagr mult of constrained pblm (λ).

Same arg't applies to Dir pblm: opt'y cond for

$$\min_u \int_\Omega \frac{1}{2}|\nabla u|^2 + f u \, dx + \frac{1}{\varepsilon}\int_{\partial\Omega} |u-\varphi|^2 \, ds$$

is $-\Delta u_\varepsilon + f = 0$ in Ω, $\frac{\partial u_\varepsilon}{\partial \nu} + \frac{2}{\varepsilon}(u_\varepsilon - \varphi) = 0$ at $\partial\Omega$.

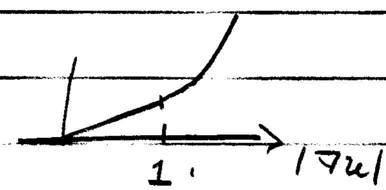
As $\varepsilon \to 0$ we expect $u_\varepsilon - \varphi \big|_{\partial\Omega} \to 0$ and

$\frac{2}{\varepsilon}(u_\varepsilon - \varphi)$ has a finite limit, which should evidently be $-\frac{\partial u_0}{\partial \nu}$ where u_0 solves the constrained problem (formally: the $\varepsilon=0$ problem).

(2). Does the minimizer of a variational pbm solve the EL eqn?

Without some conds, question might not even make sense. For example if

$$W(\nabla u) = \begin{cases} 2|\nabla u| & |\nabla u| \le 1 \\ 1+|\nabla u|^2 & |\nabla u| \ge 1 . \end{cases}$$



Then W is C^1 except at 0, but EL eqn assoc to

$$\min \int_\Omega W(\nabla u) + fu \, dx$$

is undefined at pts where $\nabla u = 0$ (formal EL is $-\operatorname{div}\left(\frac{\partial W}{\partial \nabla u}\right) + f = 0$, but when ∇u is near 0,

$$\frac{\partial W}{\partial \nabla u} = \frac{\nabla u}{|\nabla u|} \quad \text{in this example}).$$

For problems like this, convex duality provides a substitute for the EL eqn.

<u>But</u>: if W is smooth enough and has growth like $|\nabla u|^p$ at ∞, then there's no problem provided

$$\frac{\partial W}{\partial \xi} \leq C\left(|\xi|^{p-1}+1\right).$$

Under these hypotheses, the usual derivation of EL is justified (we can differentiate under the integral) since $u, v \in W^{1,p}(\Omega) \Rightarrow$

$$\int_\Omega \frac{\partial}{\partial t} W(\nabla u + t\nabla v) = \int_\Omega \left\langle \frac{\partial W}{\partial \nabla u}(\nabla u + t\nabla v), \nabla v \right\rangle$$

$$\leq \left(\int_\Omega \left|\frac{\partial W}{\partial \nabla u}(\nabla u + t\nabla v)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(\int_\Omega |\nabla u|^p\right)^{\frac{1}{p}}$$

which is controlled (indep of t) by the $W^{1,p}$ norms of u and v.

New topic: <u>convex duality</u>:

- two distinct goals
- brief discussion of linear programming
- a basic pde example of convex duality
- The Fenchel transform, and derivation of dual problem by min max $\rightarrow$ max min
- some specific, interesting examples of convex duality in pde settings

• the "calibration method" (not convex duality, but closely related)

Throughout our discn, 2 goals will be intertwined:

(1) We're often interested in the min value of a convex optzn, eg

$$\inf_{bc} \int_\Omega W(\nabla u)\, dx$$

with W convex. Upper bounds are easy (any choice of u gives one). But what about lower bounds? The convex dual provides a systematic approach.

(2) We're sometimes interested in a convex but non-smooth variational problem like

$$\min_{\substack{\int_\Omega u\, dx = 1 \\ u=0 \text{ at } \partial\Omega}} \int_\Omega |\nabla u|$$

(see later in this lecture) or else the example on pg 2.3. The convex dual provides nec + suff't cond's for optimality (playing a role analogous to

The EL eqn in the smooth case)

Many key ideas are already visible in <u>linear programming</u>. Consider (to fix ideas) the "primal problem"

$$(P) \qquad \min_{\substack{\sum_{j=1}^n a_{ij} x_j \geq b_i \\ x_j \geq 0}} \sum_{j=1}^n c_j x_j \qquad x \in \mathbb{R}^n, \quad 1 \leq i \leq m.$$

We can derive "trivial lower bd" on optimal value by taking a linear combination of the constraints: if $y_i \geq 0$ and $\sum_{i=1}^m a_{ij} y_i \leq c_j$ then

$$y_i \sum_j a_{ij} x_j \geq b_i y_i \underset{\text{adding}}{\Longrightarrow} \sum_{j=1}^n c_j x_j \geq \sum_{i=1}^m b_i y_i$$

The best "trivial lower bd" is obtained by optimizing the preceding result:

$$(D) \qquad \max_{\substack{\sum_{i=1}^m a_{ij} y_i \leq c_j \\ y_i \geq 0}} \sum_{i=1}^m b_i y_i$$

The <u>duality theorem</u> of linear programming says

$$\max \mathcal{D} = \min \mathcal{P}$$

ie the *exact* value of $\min \mathcal{P}$ is achieved by a well-chosen "trivial lower bd." (Proof of this thm is *not* trivial. See P. Lax's book on Linear Algebra for a nice proof close to the spirit of this lecture. But *any* linear programming text will give a proof; I like the text by Chvatal.)

Note: if y^* solves $\mathcal{D}$ and x^* solves $\mathcal{P}$ then (by duality thm) $\sum_j c_j x_j^* = \sum_i b_i y_i^*$. Examining calcn on prev pg we see that certain relations must hold:

$\forall i,\ y_i^* \geq 0$ and $\sum_j a_{ij} x_j^* \geq b_i$ with equality in at least one of the two

$\forall j,\ x_j^* \geq 0$ and $\sum_i a_{ij} y_i^* \leq c_i$ with equality in at least one of the two

These "*complementary slackness*" conditions play the role in linear programming that the EL eqn plays in smooth convex pde-type problems (ie: they characterize the optimal x^* — and the optimal y^*).

Here is a basic pde example of two problems in duality. Suppose $f:\partial\Omega \to \mathbb{R}$ satisfies $\int_{\partial\Omega} f\,ds = 0$. Consider

$$(P) \qquad \min_u \int_\Omega \tfrac{1}{2}|\nabla u|^2 dx - \int_{\partial\Omega} uf\,ds$$

$$(D) \qquad \max_{\substack{\operatorname{div}\sigma = 0 \text{ in } \Omega \\ \sigma\cdot n = f \text{ at } \partial\Omega}} \; -\tfrac{1}{2}\int_\Omega |\sigma|^2$$

These problems have the same relationship to each other as our P + D in Linear Programming:

① if $\left\{\begin{matrix}\sigma \text{ admissible} \\ \text{for } D\end{matrix}\right\}$ + $\left\{\begin{matrix}u \text{ admissible} \\ \text{for } P\end{matrix}\right\}$ then

value of D at σ $\le$ value of P at u

② equality holds if u solves P and σ solves D

Proof is elementary: to see ①, expand

$$\int_\Omega \tfrac{1}{2}|\sigma - \nabla u|^2 \ge 0$$

to see $\int_\Omega \tfrac{1}{2}|\sigma|^2 + \tfrac{1}{2}|\nabla u|^2 - \langle \sigma, \nabla u\rangle\,dx \ge 0$. Then use

$\operatorname{div}\sigma = 0$, $\sigma \cdot n = f$ to get

$$-\frac{1}{2}\int_\Omega |\sigma|^2 \le \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_{\partial\Omega} uf$$

To see ②, observe that if u^* solves $\mathcal{P}$ and $\sigma^* = \nabla u^*$ then preceding ineqs are $=$. So $\sigma^* = \nabla u^*$ solves $\mathcal{D}$. (It is the only soln, since $\mathcal{D}$ is strictly convex.)

Thus: in this case cond that σ optimizes $\mathcal{D}$ + u optimizes $\mathcal{P}$

$$\begin{array}{l} \operatorname{div}\sigma = 0 \text{ in } \Omega \\ \sigma \cdot n = f \text{ at } \partial\Omega \end{array} \quad + \quad \sigma = \nabla u$$

is just a rewrite of the EL eqn for $\mathcal{P}$.

<u>Digression 1</u>: I skipped over the question: what is the proper function space for $\mathcal{D}$? Everything is fine if σ is smooth; but given the form of $\mathcal{D}$, the obvious place to seek a solution (eg by the Direct Method) is

$$X = \left\{ \sigma : \int_\Omega |\sigma|^2 < \infty \text{ and } \operatorname{div}\sigma = 0 \text{ in } \Omega \right\}.$$

Does the constraint "$\sigma \cdot n = f$ at $\partial\Omega$" make sense? Answer is yes: there's a cont's map $X \to H^{-1/2}(\partial\Omega)$ taking

$$\sigma \longrightarrow \sigma\cdot n\Big|_{\partial\Omega},$$

and Green's formula holds in the sense that

$$\int_{\partial\Omega} (\sigma\cdot n)\, u = \int_{\Omega} \langle \sigma, \nabla u\rangle + \cancel{\int_{\Omega} u \operatorname{div}\sigma} \quad \text{for all } u\in H^1(\Omega).$$

Here

$$H^{-1/2}(\partial\Omega) = \text{dual of } H^{1/2}(\partial\Omega) \text{ in } L^2 \text{ inner product}$$

and

$$H^{1/2}(\partial\Omega) = \text{exact space of boundary traces of } H^1(\Omega) \text{ functions.}$$

Special case $\Omega\subset\mathbb{R}^2$ is more elementary since $\operatorname{div}\sigma = 0 \Rightarrow \sigma = (\nabla\varphi)^\perp$ and $\sigma\cdot n\big|_{\partial\Omega} = \partial_{\tan}\varphi\big|_{\partial\Omega}$.

<u>Digression 2</u>: When doing numerical calculations by finite element method it can be <u>difficult</u> to know how good an approxn you have obtained. A "primal-dual" method studies $\mathcal{P}$ + $\mathcal{D}$ simultaneously. <u>Lemma</u>: If $\hat\sigma$ is admissible for $\mathcal{D}$ + $\hat u$ is admissible for $\mathcal{P}$ and

$$(\text{value of } \mathcal{P} \text{ at } \hat u) - (\text{value of } \mathcal{D} \text{ at } \hat\sigma) < \delta$$

Then

$$\frac12\int_\Omega |\nabla\hat u - \nabla u^*|^2 < \delta \quad\text{and}\quad \frac12\int_\Omega |\hat\sigma - \sigma^*|^2 < \delta$$

where $u^* + \sigma^* = \nabla u^*$ are the solutions of $\mathcal{P}$ and $\mathcal{D}$.

(Proof: exercise. Note the similarity to Exercise 6 at the end of Lecture 1.)

<u>How to find dual problems systematically?</u>

Central idea: a convex optimization can be expressed as a minimax. Switching the min + max gives the dual plm. (There can be more than one min/max repn of a given var'l plm; different choices may lead to slightly different "dual" plms.)

More detail now, focusing on

$$(\mathcal{P}) \quad \min_{u} \int_{\Omega} W(\nabla u) - \int_{\partial\Omega} u.f$$

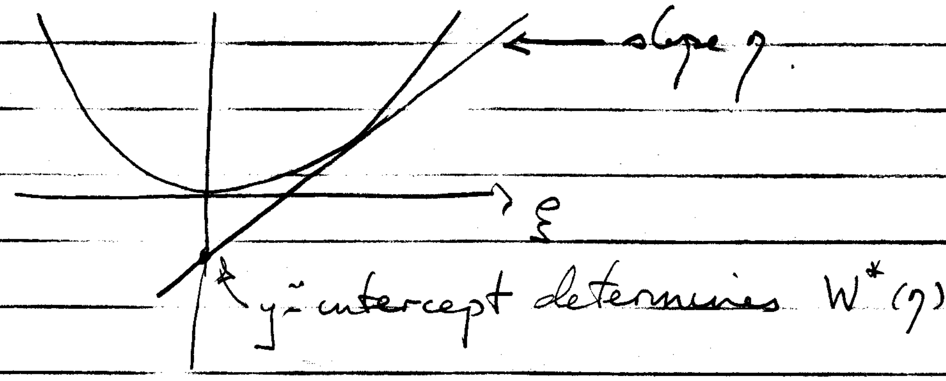
with $W(\xi)$ convex and $\int_{\partial\Omega} f\, ds = 0$.

Key pt: W convex $\iff$ its graph is the envelope of its supporting hyperplanes

$$\iff W(\xi) = \sup_{\eta} \langle \eta, \xi \rangle - W^*(\eta)$$

where W^* (the "Fenchel transform" of W) is defined by

$$W^*(\eta) = \sup_{\xi} \langle \xi, \eta \rangle - W(\xi)$$



So:

$$\min_u \int_\Omega W(\nabla u) - \int_{\partial\Omega} u \cdot f \, ds$$

$$= \min_u \max_{\sigma(x)} \int_\Omega \langle \sigma, \nabla u \rangle - W^*(\sigma) \, dx - \int_{\partial\Omega} u \cdot f$$

$$= \min_u \max_\sigma \int_{\partial\Omega} (\sigma \cdot n) u - f u \, ds - \int_\Omega (\operatorname{div} \sigma) u + W^*(\sigma) \, dx$$

<u>Claim:</u> we can switch min + max. (Return to this soon). Then above is

$$= \max_{\sigma(x)} \min_u \int_{\partial\Omega} [(\sigma \cdot n) - f] u \, ds - \int_\Omega (\operatorname{div} \sigma) u + W^*(\sigma) \, dx$$

(D)

$$= \max_{\substack{\operatorname{div} \sigma = 0 \text{ in } \Omega \\ \sigma \cdot n = f \text{ at } \partial\Omega}} - \int_\Omega W^*(\sigma) \, dx$$

since if $\operatorname{div} \sigma \neq 0$ or $\sigma \cdot n \neq f$ then min over u would be $-\infty$.

Why is min max = max min? An <u>inequality</u> is trivial, with no structural hypotheses:

<u>1st viewpt</u>: $\min_y F(x,y) \le F(x,y_0)$

$$\Rightarrow \max_x \min_y F(x,y) \le \max_x F(x,y_0).$$

$$\Rightarrow \max_x \min_y F(x,y) \le \min_y \max_x F(x,y)$$

<u>2nd viewpt</u>: if $\operatorname{div} \sigma = 0$ + $\sigma \cdot n = f$ then integration of the ptwise inequality

$$W(\nabla u) \ge \langle \nabla u, \sigma \rangle - W^*(\sigma)$$

gives $$\int_\Omega W(\nabla u) - \int_{\partial\Omega} u \cdot f \ge - \int_\Omega W^*(\sigma)$$

Either way, we see that every admissible σ for $(\mathcal{D})$ gives a lower bound on (P).

The fact that we get <u>equality</u>, ie that $\max \mathcal{D} = \min P$, is nontrivial in general. Viewed as a saddle pt principle ($\min_x \max_y F(x,y) = \max_y \min_x F(x,y)$) it requires some conds on F (typically: concave in y, convex in x, and a little more - see eg Ekeland + Temam).

But: if P or D has a clearly-defined EL eqn then it gives us a direct proof. In present setting: if W is smooth enough that

$$\operatorname{div}\left(\frac{\partial W}{\partial \nabla u}\right) = 0 \text{ in } \Omega, \quad \frac{\partial W}{\partial \nabla u}\cdot n = f \text{ at } \partial\Omega$$

at optimal u^*, then we can take $\sigma = \left.\frac{\partial W}{\partial \nabla u}\right|_{u=u^*}$.

(Note: when $W(\xi) = \frac{1}{2}|\xi|^2$, $W^*(\sigma) = \frac{1}{2}|\sigma|^2$, and this example reduces to our prior quadratic one.)

Digression: In discussing the "Direct Method" we used a theorem from functional analysis to show that

$$E[u] = \int_\Omega W(\nabla u)\, dx$$

is lower semicontinuous under weak convergence of u (if W is convex with pth power growth). The Fenchel transform gives a different, more intuitive proof of lower semicontinuity. In fact,

$$E[u] = \sup_{\sigma(x)} \int_\Omega \langle \sigma, \nabla u\rangle - W^*(\sigma)\, dx$$

and if σ is fixed then

$$u \to \int_\Omega \langle \sigma, \nabla u \rangle - W^*(\sigma)$$

is cont's under weak convergence (since it is linear in u). So $E[u] =$ max of cont's fns $\Rightarrow$ it is lower semicontinuous.

Plan for the rest of this lecture: discuss

(A) a more subtle example of duality that still involves linear pde

(B) some examples involving L^1-L^∞ duality, where there is no conventional Euler-Lagrange eqn

(C) discuss connection to the "calibration method," in context of minimal surfaces.

Here's example A: Let $\lambda_0 = 1^{st}$ Dirichlet eigenvalue of domain $\Omega \subset \mathbb{R}^n$

$$\lambda_0 = \min_{u=0 \text{ at } \partial\Omega} \frac{\int_\Omega |\nabla u|^2\, dx}{\int_\Omega u^2\, dx} = \min_{\substack{u=0 \text{ at } \partial\Omega \\ \int_\Omega u^2 = 1}} \int_\Omega |\nabla u|^2 .$$

Upper bounds are easy (consider any u). How about a scheme for proving lower bounds?

Step 1: Suff't to consider $u \geq 0$, since replacing u by $|u|$ leaves both $\int_\Omega |\nabla u|^2$ and $\int_\Omega u^2$ unchanged (Exercise)

Step 2: Let $\rho = u^2$ (ie let $u = \sqrt{\rho}$) and write defn of λ_0 in terms of ρ:

$$\lambda_0 = \min_{\substack{\int_\Omega \rho \, dx = 1 \\ \rho \geq 0 \text{ in } \Omega \\ \rho = 0 \text{ at } \partial\Omega}} \int_\Omega \frac{|\nabla \rho|^2}{4\rho} \, dx$$

Step 3: Observation: The function $\xi^2/4t$ is a convex function of (ξ, t); in fact

$$\xi^2/4t = \max_\sigma \langle \sigma, \xi \rangle - t|\sigma|^2$$

So

$$\lambda_0 = \min_{\substack{\int_\Omega \rho = 1 \\ \rho \geq 0 \\ \rho = 0 \text{ at } \partial\Omega}} \max_{\sigma(x)} \int_\Omega \langle \sigma, \nabla \rho \rangle - \rho |\sigma|^2$$

step 4 Proceed as usual: switch min + max, and use min over ρ to get constraints on σ

$$\lambda_0 \overset{(?)}{=} \max_{\sigma} \min_{\substack{\int\rho=1 \\ \rho\geq 0 \\ \rho=0 \text{ at } \partial\Omega}} \cancel{\int_{\partial\Omega} \rho\,\sigma\cdot n}^{\,0} - \int_\Omega \left[(\operatorname{div}\sigma) + |\sigma|^2\right]\rho$$

$$= \max_{-[\operatorname{div}\sigma + |\sigma|^2] \geq \mu \text{ (constant!)}} \mu$$

= largest constant μ st $\exists$ vector field σ on Ω with $\operatorname{div}\sigma + |\sigma|^2 \leq -\mu$ ptwise.

step 5 Is the max/min right? Sure! Observe that $\max_\sigma \langle\sigma, \xi\rangle - t|\sigma|^2$ is achieved when $\xi = 2t\sigma$. So best σ is $\frac{1}{2\rho}\nabla\rho$ where $\rho = u^2$ + u is 1st Dirichlet eigenfunction. Direct calcn $\Rightarrow$ this σ is admissible for dual plm and achieves its optimal value (Exercise).

Here is Example B: what can we say about

$$(*) \quad \min_\sigma \left\{ \|\sigma\|_\infty \text{ st } \operatorname{div}\sigma = 1 \text{ on } \Omega \right\}$$

where (to fix ideas) Ω is a (bounded) domain in $\mathbb{R}^2$?

[Interpretation: it's raining uniformly on Ω. How can rain flow to bdry with least possible local accumulation?]

Observation: it's equivalent to solve

$$(**) \qquad \max_{\substack{|\sigma|\le 1 \\ \operatorname{div}\sigma = \lambda \text{ (constant)}}} \lambda$$

since if λ_{max} is optimal for (**) then

$$\left.\begin{array}{l}\operatorname{div}\sigma = \lambda \text{ (const)} \\ |\sigma| \le 1\end{array}\right\} \Rightarrow \lambda \le \lambda_{max}$$

so

$$\left.\begin{array}{l}\operatorname{div}\left(\frac{\sigma}{\lambda}\right) = 1 \\ |\sigma/\lambda| \le \frac{1}{\lambda}\end{array}\right\} \Rightarrow \frac{1}{\lambda} \ge \frac{1}{\lambda_{max}}.$$

Thus $1/\lambda_{max}$ is the optimal value for (*).

We identify a dual pbm by the usual argument:

(**) is equiv to

$$(P) \qquad \max_{\substack{|\sigma|\le 1 \\ \operatorname{div}\sigma=\lambda}} \lambda = \max_{|\sigma|\le 1} \min_{\substack{u=0 \text{ at } \partial\Omega \\ \int_\Omega u=1}} -\int_\Omega \langle \sigma, \nabla u\rangle$$

(note: the min over u is $-\infty$ unless $\operatorname{div}\sigma = \lambda$ is constant, in which case it equals $-\lambda$).

Assuming max min = min max, dual is

$$\min_{\substack{u=0 \text{ at } \partial\Omega \\ \int_\Omega u = 1}} \ \max_{|\sigma|\le 1} -\int_\Omega \langle \sigma, \nabla u \rangle$$

Best σ has $-\langle \sigma, \nabla u\rangle = |\nabla u|$, so dual is

$$(\mathcal{D}) \qquad \min_{\substack{u=0 \text{ at } \partial\Omega \\ \int_\Omega u = 1}} \int_\Omega |\nabla u| \, dx .$$

Is max min = min max? An inequality is elementary as usual

$$\left.\begin{array}{l} |\sigma|\le 1, \ \operatorname{div}\sigma = \lambda \text{ (constant)} \\ u=0 \text{ at } \partial\Omega, \ \int_\Omega u = 1 \end{array}\right| \Rightarrow \underbrace{-\int_\Omega \langle \sigma, \nabla u\rangle}_{\lambda} \le \int_\Omega |\nabla u|$$

so $\max \mathcal{P} \le \min \mathcal{D}$. However equality of the optimal values is not simple to prove in this case. Moreover the optimal u is rather singular – it's the

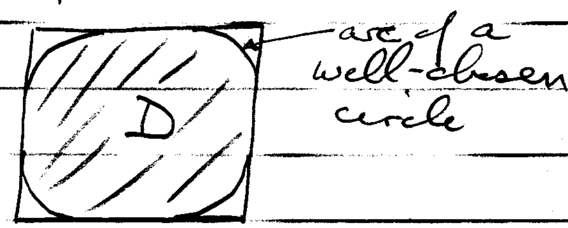
characteristic function of a set (see below).
Anyway, max min = min max is true here
(similar results are proved in Ekeland + Temam).
Analogous L^1-L^∞ duality problems arise in <u>plasticity</u>

Interesting feature of L^1-L^∞ pairs: one is typically much easier to solve than the other. In the present setting we can solve $\mathcal{D}$ more or less explicitly, using the characterization

$$(***) \qquad \min_{\substack{u=0 \text{ at } \partial\Omega \\ \int_\Omega u=1}} \int_\Omega |\nabla u| = \min_{D\subset\Omega} \frac{\text{length}(\partial D)}{\text{Area}(D)}$$

= solution of a geometry problem!

For example, if Ω = square:



(see G. Strang, "A minimax problem in plasticity theory", Springer Lecture Notes in Math 701, 1979, 319–333).

Let's sketch the proof of (***). A key ingredient is the coarea formula $\int f(x)\,|\nabla u|\,dx = \int \left(\int_{u=t} f\,ds \right) dt$

(this is easy to justify if u is nice enough – more or less it is just the "method of shells" from Calc III). Using this:

step 1 LHS of (***) $= \min\limits_{u=0 \text{ at } \partial\Omega} \dfrac{\int_\Omega |\nabla u|}{\int_\Omega u}$.

(easy)

step 2 May suppose $u \geq 0$, since replacing u by $|u|$ leaves $\int|\nabla u|$ unchanged and increases $\int_\Omega u$.

step 3 For $u \geq 0$, $\int_\Omega u\,dx = \int_0^\infty \text{Area}\{u \geq t\}\,dt$ since

$$\int_\Omega u\,dx = \int_\Omega \int_0^{u(x)} 1\,dt\,dx$$

(now use Fubini's Thm). On the other hand

$$\int_\Omega |\nabla u|\,dx = \int_0^\infty \text{length}\{u=t\}\,dt$$

(by coarea formula with $f=1$). So: if RHS of (***) has min α then

$$\text{length}\{u=t\} \geq \alpha\,\text{Area}\{u \geq t\} \text{ for all } t$$

$$\Rightarrow \int_\Omega |\nabla u| \geq \alpha \int_\Omega u.$$

This shows

$$\min_{\substack{u=0 \text{ at } \partial\Omega \\ u\geq 0}} \frac{\int_\Omega |\nabla u|}{\int_\Omega u} \geq \min_{D\subset\Omega} \frac{\text{length}(\partial D)}{\text{Area}(D)}$$

but the opposite ineq is easy (take u = char fn of D).

Digression: For discussion of closely related problems see G. Strang, "Maximum flows + minimum cuts in the plane," J Global Optzn 47 (2010) 527-535. One of the many topics there: a very efficient proof (due to Grieser) of

Cheeger's inequality: if λ_0 = 1st Dirichlet eigenvalue of Δ in Ω, and

$$h = \min_{D\subset\Omega} \frac{\text{length}(\partial D)}{\text{area}(D)}$$

Then

$$\frac{h^2}{4} \leq \lambda_0.$$

Proof: From prior discussion $\exists\, \sigma$ st $|\sigma| \leq 1$ + $\text{div}\,\sigma = h$. Let u_0 be 1st Dirichlet eigenfunction. Then

$$h\int_\Omega u_0^2 = \int_\Omega \text{div}\,\sigma \cdot u_0^2 = -2\int u_0 \langle \sigma, \nabla u_0 \rangle$$

$$\leq 2\int |u_0|\,|\nabla u_0|$$

$$\leq 2\left(\int_\Omega u_0^2\right)^{1/2}\left(\int_\Omega |\nabla u_0|^2\right)^{1/2}$$

So

$$h \leq 2\,\frac{\left(\int_\Omega |\nabla u_0|^2\right)^{1/2}}{\left(\int u_0^2\right)^{1/2}} = 2\lambda_0^{1/2}$$

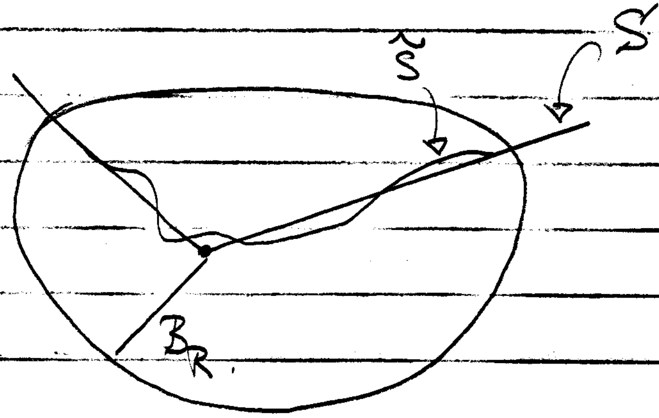
Last example (C) comes from the theory of minimal surfaces.

Some orientation: essence of duality is a scheme for proving lower bds on minimization problems (or upper bounds on maximization problems) using nothing more than integration by parts (and a well-chosen test function).

Sometimes this works even for problems that are not (in any obvious way) convex. A classic example from geometry: The 7 dimensional surface in $\mathbb{R}^8$

$$S = \{ x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2 \}.$$

(known as the Simons cone) is area-minimizing in the sense that no compactly-supported perturbation lowers the (7-dimensional) surface area. Rough sketch:



$S \cap B_R$ has minimum possible area among all surfaces with the same boundary condition at ∂B_R.

Original pf was due to Bombieri, De Giorgi, Giusti in 1969. The following much simpler pf is from G De Philippis + E Paolini, "A short proof of the minimality of Simons cone", Rend. Sem. Mat. Univ. Padova 121 (2009) 233-241.

Key idea: it's suff't to find a vector field σ in $\mathbb{R}^8$ st

$|\sigma| \leq 1$ pointwise

σ = unit normal at points on the Simons cone

$\operatorname{div} \sigma \leq 0$ "below" the Simons cone

$\operatorname{div} \sigma \geq 0$ "above" the Simons cone.

("above" means $X_5^2 + X_6^2 + X_7^2 + X_8^2 < X_1^2 + X_2^2 + X_3^2 + X_4^2$, etc)

Claim: Any such σ provides an "elementary, integration by parts based" proof that (using the notation of the previous sketch) $|\tilde{S}| \leq |S|$.

(Does σ solve some "dual problem"? No. But it gives us proof that S solves

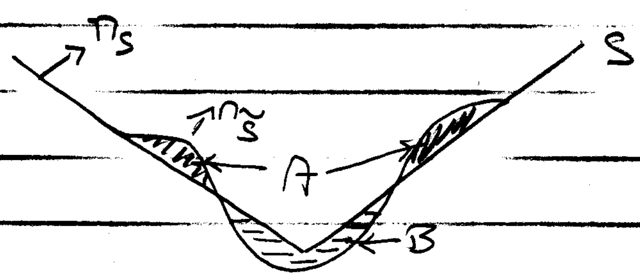
$$\min_{\tilde{S} \cap \partial B_R = S \cap \partial B_R} |\tilde{S}|$$

in much the way that soln of a convex dual problem provides proof of optimality for soln of the primal problem.)

Proof of the claim: given $\tilde{S}$, let

$$A = \{\text{pts below } \tilde{S} \text{ and above } S\}$$

$$B = \{\text{pts below } S \text{ and above } \tilde{S}\}$$

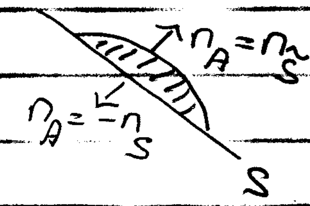


and let n_S, $n_{\tilde{S}}$ be unit normals "pointing upward".

Observe that

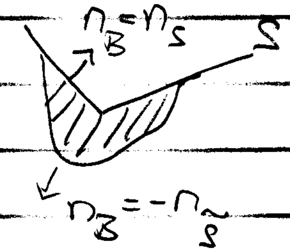
$$\int_A \operatorname{div}\sigma = \int_{\partial A} \sigma\cdot n_A$$ using the outward unit normal n_A

$$= \int_{(\partial A)\cap\tilde S} \sigma\cdot n_{\tilde S} - \int_{(\partial A)\cap S} \sigma\cdot n_S$$



$$\int_B \operatorname{div}\sigma = \int_{\partial B} \sigma\cdot n_B$$ using outward normal

$$= \int_{(\partial B)\cap S} \sigma\cdot n_S - \int_{(\partial B)\cap\tilde S} \sigma\cdot n_{\tilde S}$$



Therefore

$$\int_A \operatorname{div}\sigma - \int_B \operatorname{div}\sigma = \int_{\tilde S\cap B_R} \sigma\cdot n_{\tilde S} - \int_{S\cap B_R} \sigma\cdot n_S$$

(pts where $S=\tilde S$ aren't on either ∂A or ∂B, but such points enter into both terms on the RHS and the contributions cancel).

Now: if σ has the properties listed on pg 2.24 then LHS terms are positive, so

$$0 \le \int_A \operatorname{div}\sigma - \int_B \operatorname{div}\sigma$$

$$= \underbrace{\int_{\tilde{S}\cap B_R} \sigma\cdot n_{\tilde{S}}}_{\substack{\text{This is}\\ \le \text{ area of } \tilde{S}\\ \text{in } B_R}} - \underbrace{\int_{S\cap B_R} \sigma\cdot n_S}_{\substack{\text{This is equal}\\ \text{to area of } S\\ \text{in } B_R}}$$

So $|\tilde{S}\cap B_R| \ge |S\cap B_R|$ as asserted.

Where to find σ? It's easy: writing $z=(x_1,x_2,x_3,x_4)$ and $w=(x_5,x_6,x_7,x_8)$, let $f(z,w) = \dfrac{|z|^4-|w|^4}{4}$ and consider

$$\sigma = \frac{\nabla f}{|\nabla f|}$$

An elementary calc gives

$$|\nabla f|^3 \operatorname{div}\frac{\nabla f}{|\nabla f|} = \left(|z|^4-|w|^4\right)\underbrace{\left[3|z|^4-6|z|^2|w|^2+3|w|^2\right]}_{\text{clearly } \ge 0}$$

so div σ has the same sign as $|z|^4-|w|^4$. And the Simons cone is $\{f=0\}$, so σ points in direction of its normal.

Some exercises:

(1) On pg 2.3 we discussed $\min_u \int_\Omega W(\nabla u) + fu \, dx$ when

$$W(\xi) = \begin{cases} 2|\xi| & |\xi| \le 1 \\ 1 + |\xi|^2 & |\xi| \ge 1 \end{cases}$$

which is convex but not diff'ble at 0. What is the dual problem in this case?

(2) Show that the dual of

$$\min_{u = \varphi \text{ at } \partial\Omega} \int_\Omega \tfrac{1}{2} |\nabla u|^2$$

is

$$\max_{\operatorname{div} \sigma = 0} \int_{\partial\Omega} (\sigma \cdot n) \varphi - \tfrac{1}{2} \int_\Omega |\sigma|^2$$

(3) Show that the problems

$$(P) \qquad \min_{\substack{\operatorname{div} \sigma = F \text{ in } \Omega \\ \sigma \cdot n = f \text{ at } \partial\Omega}} \int_\Omega |\sigma|$$

(D) $\displaystyle \max_{|\nabla u|_{L^\infty} \le 1} \int_{\partial\Omega} u \cdot f \, ds - \int_\Omega u \cdot F \, dx$

are a dual pair, if $\int_\Omega F \, dx = \int_{\partial\Omega} f \, ds$. How should σ + ∇u be related if equality is to hold?

(Hint: $\displaystyle \max_{\sigma \in \mathbb{R}^n} \langle \xi, \sigma \rangle - |\sigma| = \begin{cases} 0 & \text{if } |\xi| \le 1 \\ \infty & \text{if } |\xi| > 1 \end{cases}$.)

Rmk: If $\Omega \subset \mathbb{R}^2$ and $F = 0$ then $\mathcal{P}$ can be solved explicitly in simple cases using the coarea formula. Why?

(4) Our example B concerned $\min_\sigma \{ \|\sigma\|_\infty \text{ s.t. } \operatorname{div} \sigma = 1 \text{ on } \Omega \}$. Can you do something similar for

$$\min_\sigma \left\{ \max_{x \in \Omega} \left(|\sigma_1(x)| + |\sigma_2(x)| \right) \text{ s.t. } \operatorname{div} \sigma = 1 \text{ in } \Omega \right\}?$$

(Note: a number of other "exercises" were embedded in the notes.)