Calculus of Variations, Lecture 9, 4/2/2013.

We have, up to now, focused largely on var'l pbms that have solutions; though we know from simple examples that lots of var'l pbms don't have solns (for example: $\min \int_0^1 (u_x^2-1)^2 + u^2$).

Today: some examples from applications of var'l pbms that "don't have solutions." As in the simplest 1D example (above), minimization in such settings requires "microstructure," and the natural goals are to

a) identify the favorable microstructures

b) give an algorithm for constructing minimizing sequences.

We'll achieve both (in some examples) by considering a suitable relaxed variational problem.

My examples today do not come from optimal control, but I note that there are also lots of optimal control problems that require relaxation (cf the book on reserve by Gamkrelidze).

Example 1: thin elastic sheets (reference: A C Pipkin, IMA J Appl Math 36, 1986, pp 85–99)

Consider a thin rubber sheet. To keep things simple, let's use the neoHookean model for 3D rubber, ie let's assume rubber is incompressible with "elastic energy"

$$W_{3D}(F) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3$$

where $\{\lambda_i\}_{i=1}^3$ are the prin stretches (eigenvalues of $(F^TF)^{1/2}$) and $\lambda_1\lambda_2\lambda_3 = 1$ is a constraint. This is clearly polyconvex, since $\lambda_1^2+\lambda_2^2+\lambda_3^2 = tr(F^TF)$ is a convex function of F and $\det F = 1$ is a polyconvex constraint.

Model a thin sheet by assuming the deformation gradient is constant wr to thickness, Then we can focus on the map $u: \mathbb{R}^2 \to \mathbb{R}^3$ taking the (flat) reference sheet to its position in $\mathbb{R}^3$. Since

$$|Du \cdot v|^2 = \langle Du^T Du\, v, v\rangle \quad \text{for } v \in \mathbb{R}^2,$$

stretching + compression in-plane are assoc to eigenvalues of $(Du)^T Du$: call these λ_1, λ_2. By incompressibility of the 3D sheet, the energy per unit area (per unit thickness) is thus

$$W_{2D} = \lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2\lambda_2^2} - 3.$$

The var'l pbm

$$\int_\Omega W_{2D}(Du)\,dx$$

is "in need of relaxation", in the sense that

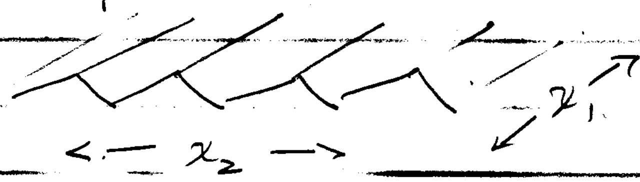
a) $\int_\Omega W_{2D}(Du)\,dx$ is not lower semicontinuous

b) for some choices of bc (or lower order terms), this var'l pbm (or its analogues with lower order terms) don't achieve their minima.

The key pt: a sheet can avoid compression by <u>wrinkling</u>. If the bc or lower order term requires the wrinkling to be infinitesimal then one gets nonexistence.

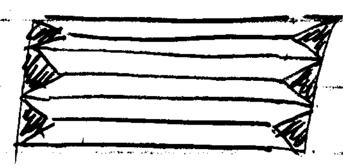
<u>A little detail</u>: consider bc $u_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ at $\partial\Omega$,

with Ω = a square in $\mathbb{R}^2$. The sheet could meet this bc using $u = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ But this involves a lot of compression. It can achieve energy 0 and almost meet the bc by wrinkling, with wrinkles parallel to the x_1 axis. Making a choice, u can even be piecewise linear



side view

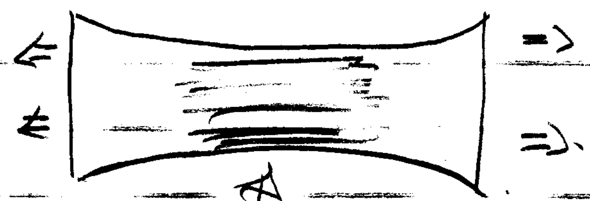
But: a wrinkled def cannot meet the bc. So one needs a bdry layer near the sides where the wrinkles end. The def can still be piecewise linear



If scale of wrinkling is δ then bdry layer of width abt δ achieves energy $\sim \delta^2$ while exactly meeting bc.

Even without bdry layer, the wrinkling arg't above gives (as $\delta \to 0$) a counterexample to lsc.

Does this mean thin elastic sheets always wrinkle? Of course not, but they can wrinkle. Classic example (Audoly + Boudaoud, PRL 2003): a rectangular sheet, pulled by stiff ruler at 2 opposite sides



wrinkles near middle

Example 2: Optimal designs (one of many pbms addressed by my work with G Strang in the 1980's; for this one see our article "Fibered structures in optimal design," in Sleeman + Jarvis eds, Theory of Ordinary + Partial Diff'l Eqns, Longman, 1986)

We discussed earlier this semester how to find the smallest t_0 st

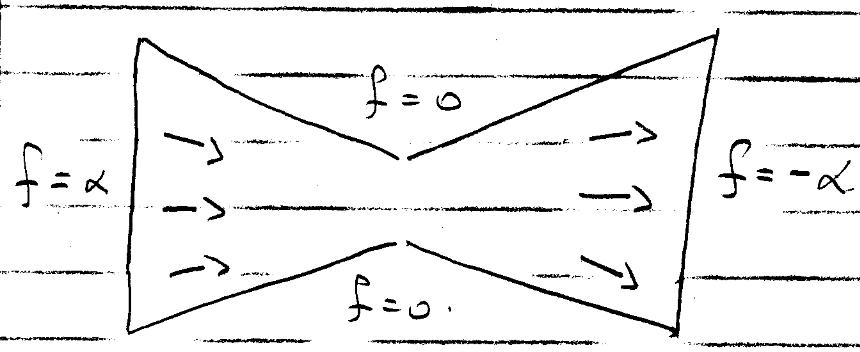
$$\exists\, \sigma,\ |\sigma| \le t_0,\ \operatorname{div} \sigma = 0 \text{ in } \Omega$$
$$\sigma \cdot n = f \text{ at } \partial\Omega$$

where $f : \partial\Omega \to \mathbb{R}$ is specified with mean value 0. (Well: we discussed similar problems, and the method we used there can be used here as well.)

Suppose the smallest t_0 is larger than 1 — so there exists σ (probably lots of them) with

$$(*) \qquad |\sigma| \le 1, \quad \operatorname{div} \sigma = 0 \text{ in } \Omega$$
$$\sigma \cdot n = f \text{ at } \partial\Omega$$

Example to keep in mind



(neck is a bottleneck for getting stuff from "source" at LHS to "sink" at RHS; need α suff'tly small for existence of such σ)

[If you don't like thinking abt div-free vector fields, question can be reformulated in 2D to involve gradients since $\operatorname{div} \sigma = 0 \iff \sigma = (\partial_2 u, -\partial_1 u)$ and $\sigma \cdot n = \partial_{\tan} u$; so (*) is equiv to

$$|\nabla u| \leq 1 \text{ in } \Omega, \qquad u = F \text{ at } \partial\Omega$$

where F = 1st integral of f.]

OK, here's the question: suppose we'd like to take $\sigma = 0$ on some subset of Ω. How big can this subset be? (If σ represents a flow, we'd like to restrict the flow to a subset of Ω; how much area can we remove?)

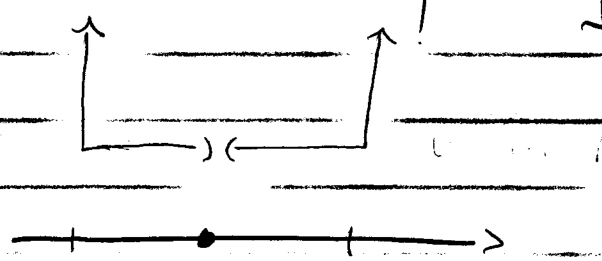
Answer: max area where $\sigma = 0 \iff$ min area where $\sigma \neq 0 \iff$

$$\min_{\substack{\operatorname{div}\sigma = 0 \text{ in } \Omega \\ \sigma \cdot n = f \text{ at } \partial\Omega \\ |\sigma| \leq 1 \text{ ptwise}}} \int_\Omega 1_{\{\sigma \neq 0\}} \, dx$$

or equivalently

$$\min_{\substack{u = F \text{ at } \partial\Omega \\ |\nabla u| \leq 1 \text{ ptwise}}} \int_\Omega 1_{\{\nabla u \neq 0\}} \, dx$$

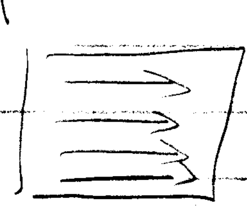
The integrand here is clearly very nonconvex



We'll show (perhaps next wk) that the answer involves solving the <u>relaxed</u> (in this case, <u>convexified</u>) problem

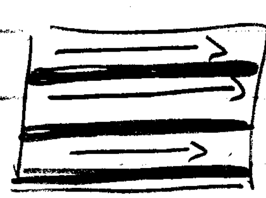
$$\min_{\substack{\sigma\cdot n=f \\ |\sigma|\le 1 \\ \operatorname{div}\sigma=0}} \int_\Omega |\sigma| \qquad \text{or} \qquad \min_{\substack{u=F \text{ at } \partial\Omega \\ |\nabla u|\le 1}} \int_\Omega |\nabla u|$$

Where "relaxed pblm" has $0<|\sigma|<1$, optimal design uses a flow with same direction, but oscillating btwn 0 + norm 1



uniform flow, magnitude less than 1

corresp to



holes parallel to flow, $|\sigma|=1$ where you keep material.

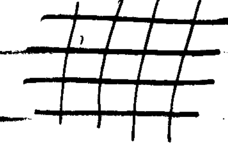
(Rmk: the relaxed pbm in this example can be solved more or less explicitly, using co-area formula. See R Kohn + G Strang, "The constrained least gradient problem" in Kropp + Lacey eds, Nonclassical Continuum Mechanics, CUP 1987, 226-243 + later, more rigorous treatment by Sternberg, Williams, + Ziemer, Trans AMS 339 (1993) 403-432.)

Example 3: Twinning due to martensitic phase transformation. (Reference: book by K, Bhattacharya on shape memory materials; or Ball + James, Phil Trans: Phys Sci + Eng'g 338, 1992, 389-450)

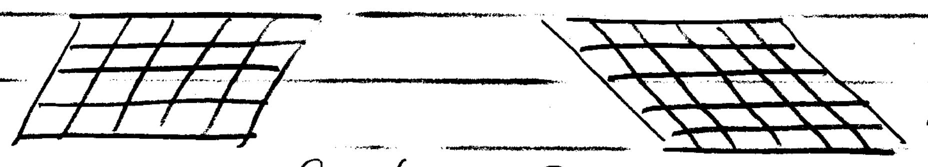
Real phenomenon is 3D but I'll explain in 2D using a "square-to-parallelogram" phase transition.

A "shape memory material" is a crystalline material with a simple (high-symmetry) structure at high temp ($T > T_c$) + several (symmetry-related) possible structures at lower temp ($T < T_c$).

2D version: cubic structure $T > T_c$



2 phases (unit cell = parallelogram) for $T < T_c$



both sheared rel to cubic phase

If we ignore issue of nucleation / motion of phase bdries, we can suppose the material chooses the phase of lowest energy for a given deformation gradient. So

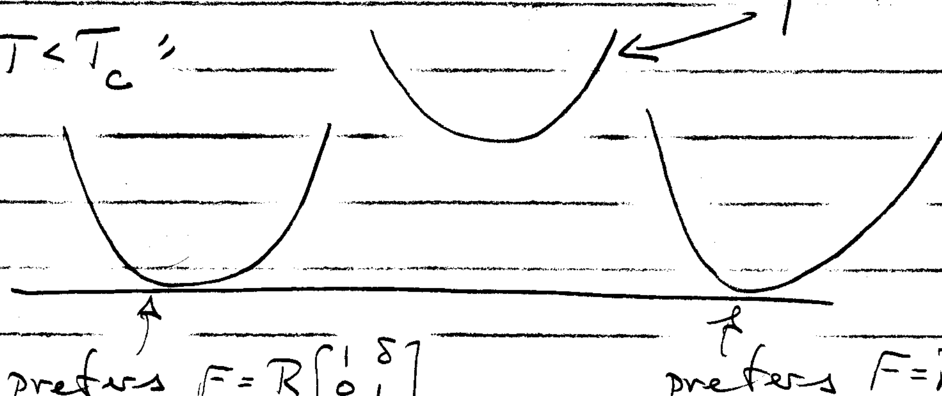
$$E = \int W(F) \qquad F = \frac{\partial x}{\partial X}$$

where

$$W = \min_{i=1,2,3} \left\{ W_i(F) \right\}$$

Picture at $T < T_c$:

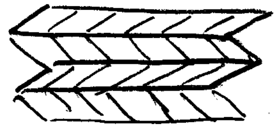
prefers $F \in SO(2)$



prefers $F = R\begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix}$ for $R \in SO(2)$

prefers $F = R\begin{bmatrix} 1 & -\delta \\ 0 & 1 \end{bmatrix}$ for $R \in SO(2)$.

The sheared phases can be <u>mixed in layers</u>



"atomic scale picture"
assoc piecewise linear + cont's deformation $x(X)$ st $F = \frac{\partial x}{\partial X}$ takes just 2 values $\begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix}$ + $\begin{bmatrix} 1 & -\delta \\ 0 & 1 \end{bmatrix}$

Average F can be $\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$ for any $|\lambda| < \delta$.

Consequence: a region of "parallelogram phase" can be surrounded by "square phase" at $T = T_c$, with energy ≈ 0, by using infinitesimal layering (as above) with $\lambda = \frac{1}{2}$. <u>This is what happens</u>.

Another question: what <u>other</u> avg defs can be achieved by mixing defs in the "wells" $SO(2)\begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix}$ + $SO(2)\begin{bmatrix} 1 & -\delta \\ 0 & 1 \end{bmatrix}$, with (presumably) more complicated geometry? Answer was determined by Ball + James in article cited above.

Wrapping around all 3 examples:

a) when there's a problem with lower semicontinuity this can often be detected very simply, using piecewise linear defs in a simple "layered" geometry.

[But; not always! — cf Sverak's pf that rank one convexity is not equivalent to quasiconvexity.]

b) applications often lead us to consider energies that are <u>very</u> nonconvex, and definitely <u>not</u> lower semicontinuous.

c) proper goal in such settings is to understand energetically prefered microstructures, and/or to give examples of energy-minimizing sequences.

d) in most settings there are natural "regularizations" that restore existence (eg "bending energy" for thin sheets, or "perimeter" for the optimal design problem).