# Calculus of Variations - Lecture 14 - 5/7/2013

Today's topic: length scale + pattern selection in var'l pbms where "energy min'zn requires microstructure".

Here's the "big picture." Depending on details such as lower-order terms + bdry conds, a nonconvex var'l pbm may have many solns or no soln.

If there are many solns, a regularizing singular perturbation will select between them, eg

- $$\min_{u(0)=u(1)=0} \int_0^1 (u_x^2-1)^2 + \varepsilon^2 u_{xx}^2$$

  has lots of solns for $\varepsilon=0$; The ones with "a single transition" are preferred when $\varepsilon>0$

- modelling a rubber band as an "elastic string" would lead to $\int_0^L W_{1D}(|\gamma'(s)|)\,ds$, where $W$ prefers $|\gamma'(s)|=1$. Inclusion of bending resistance gives instead (more or less)

  $$\min \int_0^1 W_{1D}(|\gamma'(s)|) + \varepsilon|\gamma''(s)|^2$$

and leads to a *circular* configuration.

Our focus today is instead on the case when there is *no* soln. Then the regularized pbms must "develop microstructure" as $\varepsilon \to 0$

- $\min \int_0^1 (u_x^2-1)^2 + \lambda u^2 + \varepsilon^2 u_{xx}^2$

By Modica-Mortola, expect it to behave like $\frac{8}{3}\varepsilon\cdot(\#\text{teeth}) + \int_0^1 \lambda u^2$ restr to $u_x = \pm 1$.

N teeth equally spaced $\Rightarrow$ value is about

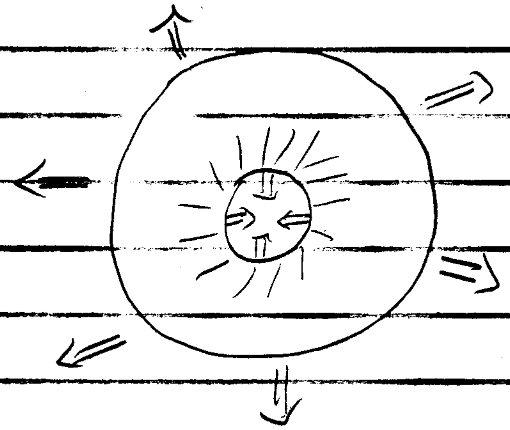
$$\frac{8\varepsilon}{3} N + C_1 \lambda \cdot \frac{1}{N^2}$$

$\Rightarrow$ best $N \approx C_2 \left(\frac{\lambda}{\varepsilon}\right)^{1/3}$ $\Rightarrow$ opt'l value $\sim \varepsilon^{2/3} \lambda^{1/3}$

One can show that this scaling is optimal (non-equispaced teeth can't do better).

[1D problems are special — by using EL eqns + 1D methods, one can show that minimizer is *exactly* periodic when $\varepsilon$ is sufftly small, cf S. Muller, Calc Var PDE 1, 1993.]

- 2D sheets under tension often "develop microstructure" as bending resistance $\to 0$. Example from Peter Bella's recent thesis work: a planar annulus



$$\int W_{2D}(Du) + \varepsilon^2 |D^2 u|^2$$

+ bdry terms for loads

$W_{2D}$ prefers $Du^T Du = I$.

inner circle "shrinks" — sheet wrinkles (while staying almost in-plane) to avoid compression.

<u>Question</u>: can we quantify how the sing perturbation influences the character of the microstructure?

A "classical" viewpt: study the bifurcation diagram. When $\varepsilon$ is large soln will be unique (pbm is practically convex). At crit $\varepsilon_*$ there's a 1st bifurcation. Below that there are further bifurcations. <u>Very</u> difficult to pursue this in practice, since bifn diagram is increasingly complex as $\varepsilon \to 0$.

Alternative viewpt: let $E_\varepsilon$ = min value of energy as fn of $\varepsilon$.

We expect that

$$\lim_{\varepsilon \to 0} E_\varepsilon = E_0 \quad \text{is min value of } \underline{\text{relaxed problem}},$$

so it makes sense to ask what is the leading-order correction:

$$\text{is } E_\varepsilon \sim E_0 + C\varepsilon^\gamma \; ?$$

In course of proving such a result, we expect to learn something about spatial structure of minimizers. For example, if

$$\int_0^1 (u_x^2 - 1)^2 + \lambda u^2 + \varepsilon^2 u_{xx}^2 \sim C \varepsilon^{2/3} \lambda^{1/3}$$

Then $\varepsilon \cdot \#(\text{teeth}) \sim \varepsilon^{2/3} \lambda^{1/3} \Rightarrow \#(\text{teeth}) \sim (\lambda/\varepsilon)^{1/3}$

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Focus today on a particular problem of this type, where we understand quite a bit. Sources:

a) RV Kohn + S Müller, Rend Sem Mat Fis Milano 62 (1992) 89-113 (an expository article, at about the level of this lecture)

b) RV Kohn + S Müller, CPAM 47 (1994) 405-435, gives full technical details

c) S. Conti, CPAM 53 (2000) 1448-1474, further development, mainly proving that minimizers are in a certain sense selfsimilar.

d) RV Kohn + S Müller, Phil Mag A 66 (1992) 697-715 emphasizes the physical setting (twinning of martensite)

<u>Here's the problem</u>, in its simplest form:

$$(*) \qquad \min_{\substack{u=0 \text{ at } x=0 \\ |u_y| = \pm 1}} \int_{[0,L]\times[0,1]} u_x^2 + \varepsilon |u_{yy}|.$$

Note the close relationship to

$$(**) \qquad \min_{u=0 \text{ at } x=0} \int_{[0,L]\times[0,1]} u_x^2 + (u_y^2-1)^2 + \varepsilon^2 |\nabla\nabla u|^2$$

(via the Modica-Mortola-like reduction $\int (u_y^2-1)^2 + \varepsilon^2 u_{yy}^2 \sim c_0 \varepsilon \int |u_{yy}|$ restricted to $u_y = \pm 1$)

When $\varepsilon = 0$ the min is assoc to a relaxed (convexified!) problem

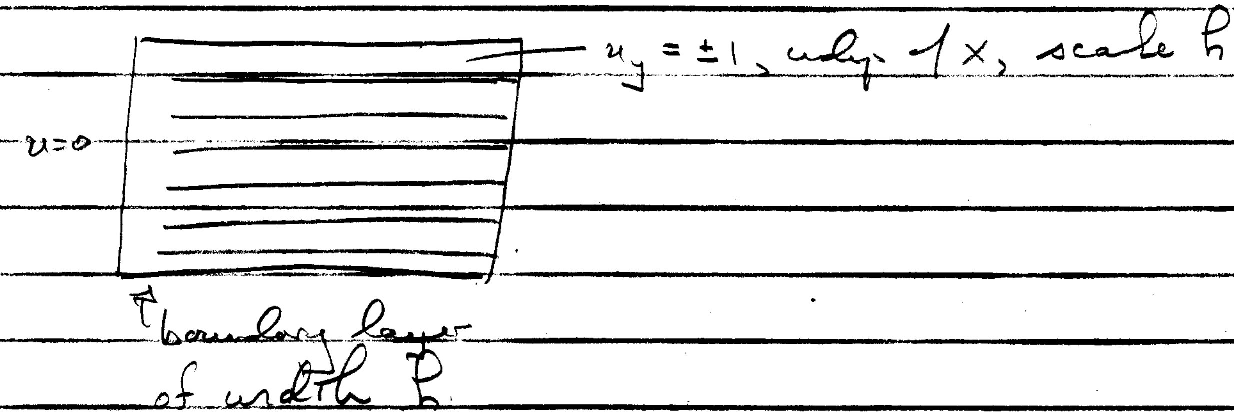
$$\text{relxn of } (*) : \quad \min_{\substack{|u_y| \le 1 \\ u=0 \text{ at } x=0}} \int_{[0,L]\times[0,1]} u_x^2$$

whose min value is 0 (achieved at $u \equiv 0$).

For pos $\varepsilon$, we therefore expect the min value $E_\varepsilon$ to scale with $\varepsilon$:

$$E_\varepsilon \sim C\varepsilon^\gamma \quad ?$$

Any constrn gives an upper bd on $E_\varepsilon$. The most obvious one



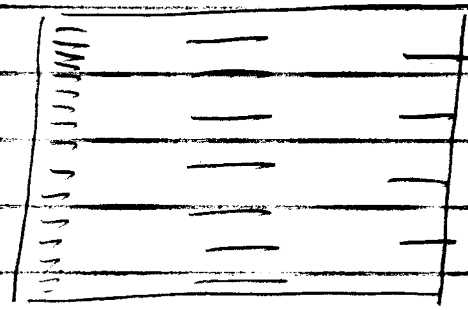

suggests (for (**), since it's hard to do this with $u_y = \pm 1$) $E_\varepsilon \sim \min_h \; h + \varepsilon L/h \sim \varepsilon^{1/2} L^{1/2}$;

But this is suboptimal. One can do better by letting the scale change with $x$. If local scale is $h(x)$ then

$$u_x \sim h_x \quad , \quad \int_0^1 |u_{yy}|\, dy \sim \frac{1}{h(x)}$$

$\Rightarrow$ energy is like $\int_0^L h_x^2 + \frac{\varepsilon}{h(x)}\, dx$. The terms (approx) balance when $h(x) = \varepsilon^{1/3} x^{2/3}$, leading to

$$E_\varepsilon \sim \varepsilon^{2/3} L^{1/3}$$



Very logical: bc needs fine structure ($h=0$) but regularization prefers coarser structure. Changing scale costs energy ($u_x \neq 0$) but may still be worthwhile.

Constrn of type indicated can be done – it isn't very hard – just specify the jump set (where $u_y$ changes) then determine $u$ by integn in $y$.

More subtle: is a better scaling possible?
Ans is no. Here's the proof:

<u>step ①</u> Let $E_\varepsilon$ be min energy. Then for some $x_0$, $L/2 < x_0 < L$, # jumps at $x = x_0$ is at most $N$ where

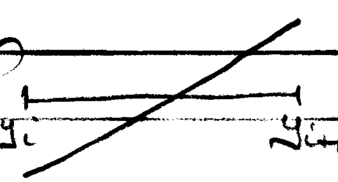
$$2\varepsilon N = \varepsilon \int_0^1 |u_{yy}|\, dy \le \frac{E_\varepsilon}{L/2} = \frac{2}{L} E_\varepsilon .$$

<u>step ②</u> Consider $y \to u(x_0, y)$. It has $u_y = \pm 1$ with $N$ "teeth". So its $L^2$ norm cannot be too small. In fact

$$\int_0^1 u^2\, dy \ge \text{const} \cdot \frac{1}{N^2}$$

Pf: let $y_1, \dots, y_N$ be pts where $u_y$ changes sign. Then

$$\int_{y_i}^{y_{i+1}} u^2\, dy \ge$$ value achieved when $u = 0$ at midpt.




$= \text{const}\, (\Delta y)^3$ with $\Delta_i y = y_{i+1} - y_i$

By Jensen (since $t^3$ is convex)

$$\frac{1}{N^3} = \left( \frac{1}{N} \sum_i \Delta_i y \right)^3 \le \frac{1}{N} \sum_i (\Delta_i y)^3$$

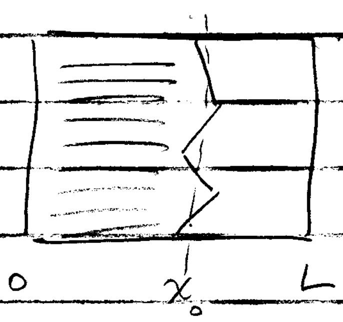
so $\int_0^1 u^2\,dy \geq \text{const}\cdot\sum{}'(\Delta y)^3 \geq \frac{\text{const}}{N^2}$

(note: all consts here are computable!)

step ③ Summary so far: $E_\varepsilon$ small $\Rightarrow$ $N$ small (at some $x_0$) $\Rightarrow$ $\int_0^1 u^2\,dy$ large (at $x_0$).

But then

$$\iint_{\substack{0<x<x_0\\0<y<1}} u_x^2 + \varepsilon|u_{yy}|\,dx\,dy \geq \min_{\substack{u=0 \text{ at } x=0\\ u(x_0,y) \text{ given at } x=x_0}} \int_0^{x_0}\int_0^1 u_x^2\,dx$$



essentially, the <u>relaxed</u> problem with bc at both $x=0$ + $x=x_0$.

This "relaxed" var pbm is convex, + EL eqn is $u_{xx}=0$, so minimizer $v$ is linear interpolant:

$$v(x,y) = \frac{x}{x_0}\,u(x_0,y)$$

for which $v_x = \dfrac{u(x_0,y)}{x_0}$, $\displaystyle\iint_{\substack{0<x<x_0\\0<y<1}} v_x^2\,dx\,dy = \frac{1}{x_0}\int_0^1 u^2(x_0,y)\,dy.$

In particular (since $X_0 \sim L$)

$$E_\varepsilon \geq \text{const} \cdot \frac{1}{L} \cdot \frac{1}{N^2}$$

Summary: small "surface energy" => large oscillations => large $u_x^2$

step ④ Put this together: writing step 1 as

$$N^{-1} \geq \frac{\varepsilon L}{E_\varepsilon}$$

we have shown

$$E_\varepsilon \geq \text{const} \, \frac{1}{L} \left( \frac{\varepsilon L}{E_\varepsilon} \right)^2.$$

$$\Rightarrow \quad E_\varepsilon \geq \text{const} \cdot \varepsilon^{2/3} L^{1/3} \qquad \text{as asserted}$$

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Does constrn have to resemble ours, to get this scaling law? Certainly not, if we don't pay attn to prefactors. For example, refining again at $x = L$ (so $u = 0$ at both $x = 0$ + $x = L$) would achieve same scaling law (with about double the prefactor).

Well, does the minimizer have to resemble our constrn? Ans is yes, more or less.

One result of this type: if $u_\varepsilon$ is the minimizer of the $\varepsilon$ plm, then for each $l \in (0,L)$

$$\int_0^1\int_0^l u_x^2 + \varepsilon|u_{yy}|\,dx\,dy \le C_1 \varepsilon^{2/3} l^{1/3};$$

also

$$\int_{x=l} u_x^2 + \varepsilon|u_{yy}|\,dy \ge C_2\,\varepsilon^{2/3} l^{-2/3}.$$

In short: the spatial distrn of energy agrees with our construction.

Main ingredients

(A) Replacing $u_\varepsilon$ by some construction done by hand for $x<l$ must increase the energy.

Since relaxed plm has min $= \frac{1}{l}\int_0^1 u^2(x_0,y)\,dy$ for energy btwn $0$ + $l$ (recall step 3 above) a good constrn should do only a little worse. By this method one proves, for any $\delta>0$,

$$\int_0^1\int_0^l u_x^2 + \varepsilon|u_{yy}| \le \frac{1+\delta}{l}\int_0^1 u^2(l,y)\,dy + C_\delta\,\varepsilon^{2/3} l^{1/3}$$

where $u = u_\varepsilon$ is the minimizer.

Now note that $u(l,y) = \int_0^l u_x\,dx \le \left(\int_0^l u_x^2\,dx\right)^{1/2} l^{1/2}$, so

$$\frac{1}{l}\int_0^1 u^2(l,y)\,dy \le \int_0^1\int_0^l u_x^2\,dx\,dy$$

Combining prev two ests:

$$(***) \quad \int_0^1\int_0^l u_x^2 + \varepsilon|u_{yy}| \le (1+\delta)\int_0^1\int_0^l u_x^2\,dx\,dy + C\varepsilon^{2/3} l^{1/3}$$

<u>Problem:</u> $u_x^2$ on LHS has coeff't 1, on RHS has coeff't $1+\delta$.

<u>Soln:</u> actually we expect $u_x^2 + \varepsilon|u_{yy}|$ to be comparable, so LHS is really like $2\iint u_x^2$.

<u>Explanation:</u> For our constrained prbm (with $u_y = \pm 1$) pf is a bit technical, but for

$$\iint u_x^2 + (u_y^2-1)^2 + \varepsilon^2 u_{yy}^2 \quad dx\,dy$$

It's elementary: just take EL eqn, mult by $u_x$, integrate in $y$ + integrate by pts to get

$$\frac{d}{dx}\int_0^1 u_x^2 - \left((u_y^2-1)^2 + \varepsilon^2 u_{yy}^2\right) dy = 0.$$

(Exercise!). For constrained pbm, analogous assertion is

$$\int_0^1 u_x^2 - \varepsilon |u_{yy}| \, dy = \text{const, indep of } x$$

Call this const $c_0$. Recalling that

$$\int_0^1\int_0^L u_x^2 \, dx\,dy \le C\varepsilon^{2/3} L^{1/3}$$

$$\int_0^1\int_0^L \varepsilon |u_{yy}| \, dx\,dy \le C\varepsilon^{2/3} L^{1/3}$$

we see that $c_0 \le C\varepsilon^{2/3} L^{-2/3}$. So

$$\int_0^1\int_0^l u_x^2 \, dx\,dy \le \int_0^1\int_0^l \varepsilon |u_{yy}| \, dx\,dy + c_0 l.$$

$$\le \int_0^1\int_0^l \varepsilon |u_{yy}| \, dx\,dy + C\varepsilon^{2/3} l^{1/3}$$

since $l \le L$

Thus (***) becomes

$$\int_0^1\int_0^l u_x^2 + \varepsilon|u_{yy}| \le \frac{1+\delta}{2}\int_0^1\int_0^l u_x^2 + \frac{1+\delta}{2}\int_0^1\int_0^l \varepsilon|u_{yy}| + C\varepsilon^{2/3} l^{1/3}$$

$$\Rightarrow \int_0^1\int_0^l u_x^2 + \varepsilon|u_{yy}| \le \tilde{C}\varepsilon^{2/3} l^{1/3}$$

Given this, a lower bd on $\int_0^1 u_x^2 + \varepsilon |u_{yy}| \, dy$ at $x = \ell$ is easy. In fact, reviewing our first argument ("steps 1 - 4") we showed

$$\begin{aligned} \text{small } \int_0^1 \varepsilon |u_{yy}| \text{ at } x=\ell &\Rightarrow \text{few teeth at } x=\ell \\ &\Rightarrow \text{large } L^2 \text{ norm at } x=\ell \\ &\Rightarrow \text{large } \int_0^1\int_0^\ell u_x^2 \, dx\,dy. \end{aligned}$$

Since we have an upper bd on $\int_0^1\int_0^\ell u_x^2 \, dx\,dy$, we can use this arg't to show

$$\int_0^1 \varepsilon |u_{yy}| \, dy \text{ cannot be too small at } x = \ell.$$

Carrying out the steps, this arg't gives

$$\int_0^1 \varepsilon |u_{yy}| \, dy \geq C \varepsilon^{2/3} \ell^{-2/3}$$

which is actually stronger than the assertion on pg (14.10).

Perspective:

(1) Our arg't "few teeth $\Rightarrow$ large $L^2$ norm" was very one-dimensional, however this result

has natural multidim'l extensions. This is best seen by writing it as

$$\|f'\|_{L^{4/3}} \le C\left(\int |f''|\right)^{1/2} \|f\|_{L^2}^{1/2} \quad \text{if } \int_0^1 f = 0$$

applied when $f' = \pm 1$. We see this way that the essence of the matter is use of interpolation inequalities relating higher norms (those in the regularization) + lower norms

(2) Identifn of "optimal scaling law" – analogous to

$$E_\varepsilon \sim \varepsilon^{2/3} L^{1/3}$$

proved earlier – has been achieved for quite a few pbms. But more local results (analogous to $\int_0^1 \int_0^\ell u_x^2 + \varepsilon |u_{yy}| \sim \varepsilon^{2/3} \ell^{1/3}$ for a minimizer) are still very elusive (i.e. still open in most settings).