Calculus of Variations, Lecture 13, 4/30/2013

Today's goals:

(1) Sketch of proofs assoc the Modica-Mortola example

(2) Elastic strings + membranes, as another example of Γ-convergence

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Recall from Lecture 12: key assertions assoc Modica-Mortola (from perspective of Γ-convergence) are as follows: if $\Omega \subset \mathbb{R}^n$ is bdd,

$$F_\varepsilon(u) = \int_\Omega \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon}(u^2-1)^2$$

$$F_0(u) = \begin{cases} 8/3 \cdot \mathrm{Per}_\Omega \{x : u(x) = 1\} & \text{if } u = \pm 1 \text{ ae} \\ \infty & \text{otherwise} \end{cases}$$

Then

① if $v_\varepsilon \to v_0$ in $L^1(\Omega)$ as $\varepsilon \to 0$,

$$\liminf F_\varepsilon(v_\varepsilon) \geq F_0(v_0)$$

② if $v_0 \in L^1(\Omega)$ then $\exists\, v_\varepsilon$ st $v_\varepsilon \to v_0$ in $L^1(\Omega)$ and

$$F_\varepsilon(v_\varepsilon) \to F_0(v_0)$$

[Taken together, these say – by defn – that F_ε Γ-converges to F_0 wrt the $L^1(\Omega)$ topology.]

Also, the functionals F_ε are in a certain sense uniformly coercive, more precisely boundedness of F_ε implies compactness in L^1:

③ if $\{v_\varepsilon\}$ has $F_\varepsilon(v_\varepsilon)$ unif bdd, then $\{v_\varepsilon\}$ is compact in L^1 (ie each sequence has a convergent subsequence).

We've already seen all the essential ingredients of the proofs, but let's discuss now how one fills in the details.

Arg't uses some basic facts about the function space

$$BV(\Omega) = \{ u \in L^1(\Omega) \text{ st } \int_\Omega |\nabla u| < \infty \}$$

where we use the (slightly abusive) notation

$$\int_\Omega |\nabla u| = \sup_{\substack{g \in C_0^\infty(\Omega, \mathbb{R}^n) \\ |g| \le 1 \text{ ptwise}}} \int_\Omega u \operatorname{div} g \, dx$$

Here $|\nabla u|$ is in general not an L^1 function, but rather a measure, which acts on cont's fns by

$$\int_\Omega h\,|\nabla u| = \sup_{\substack{g\in C_0^\infty(\Omega,\mathbb{R}^n)\\ |g(x)|\le h(x) \text{ ptwise}}} \int_\Omega u \operatorname{div} g .$$

Essentially: $u\in BV(\Omega) \iff \nabla u$ is a vector valued measure on Ω, with finite total variation $\int|\nabla u|$.

Note that by Green's formula, the char fn of a set A with smooth bdry is in BV, and

$$\operatorname{Per}_\Omega(A) = \int_\Omega |\nabla \chi_A| = \text{Surface area measure of } (\partial A)\cap\Omega$$

(Exercise: prove this, using Green's formula.)

Some (relatively easy) facts:

A) Lower semicontinuity: if $u_\varepsilon \to u$ in $L^1(\Omega)$ Then

$$\liminf_{\varepsilon\to 0}\int_\Omega |\nabla u_\varepsilon| \ge \int_\Omega |\nabla u|$$

(This is obvious, since $\int|\nabla u|$ is a sup of cont's linear functionals on L^1.)

B) Bounded sets in the BV norm
($\|u\|_{BV} = \|u\|_{L^1} + \int|\nabla u|$) are compact in L^1

(This is the BV analogue of the familiar stmt that the embedding $W^{1,p} \hookrightarrow L^q$ is compact for $\Omega \subset \mathbb{R}^n$ bdd when $q < \frac{np}{n-p}$. Our case is $q=1$ and [roughly] $p=1$. Our proof starts by showing that

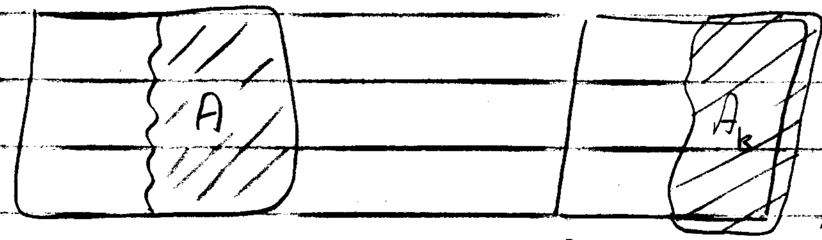
$$\inf_{I\in\mathbb{R}} \int_Q |u-I| \le C\int_Q |\nabla u| \qquad \text{when } Q=[0,1]^n,$$

which rescales to

$$\inf_{I\in\mathbb{R}} \int_{Q_\ell} |u-I| \le C\ell\int_{Q_\ell} |\nabla u| \qquad \text{when } Q=[0,\ell]^n$$

It follows (at least if Ω is a union of cubes) that $u\in BV(\Omega) \Rightarrow u$ can be approx well in L^1 by a piecewise constant function. The asserted compactness is relatively easy to establish, based on this fact.

C) If a set A has bounded perimeter (ie its char fn $\chi_A \in BV$) then there are sets $A_k \subset \mathbb{R}^n$ with C^2 bdry st $\chi_{A_k} \to \chi_A$ in L^1 and $\mathrm{Per}_\Omega(A_k) \to \mathrm{Per}_\Omega(A)$ and $\mathcal{H}^{n-1}(\partial A_k \cap \partial\Omega)=0$



Ω = square
A = shaded

$\partial A_k \cap \Omega$ is C^2
$Per_\Omega(A_k) \simeq Per_\Omega(A)$
∂A_k avoids $\partial\Omega$

(note that taking $A_k \subset \Omega$ is <u>not</u> possible)

(Proof of (C) uses mollification combined with the co-area formula.)

The book by Jost + Li-Jost has complete discussions of (A) + (B) and all the ingredients needed to prove (C).

Accepting (A)-(C) as above, we now discuss the Γ-convergence (assertions ① + ②) and compactness (assertion ③):

<u>Proof of ①</u>. Observe first that it suffices to consider limits u st $u(x) = \pm 1$ ae since if $u_\varepsilon \to u$ in L^1 then

$$\liminf_\varepsilon \int \varepsilon|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}(u_\varepsilon^2-1)^2 \geq \liminf_\varepsilon \frac{1}{\varepsilon}\int (u_\varepsilon^2-1)^2$$

$$\geq \int \liminf_\varepsilon \frac{1}{\varepsilon}(u_\varepsilon^2-1)^2$$

$= \infty$ if u takes values other than ± 1 on a set of pos measure

(we used Fatou's lemma above to move lim inf inside the integral).

Next: observe it suffices to consider u_ε st

$$-1 \leq u_\varepsilon \leq +1$$

since otherwise we can replace u_ε by its truncation

$$u_\varepsilon^* = \begin{cases} +1 & u_\varepsilon(x) > 1 \\ u_\varepsilon(x) & -1 \leq u_\varepsilon(x) \leq +1 . \\ -1 & u_\varepsilon(x) < 1 \end{cases}$$

without changing the L^1 limit (This truncation decreases the value of F_ε).

Finally, suppose $-1 \leq u_\varepsilon(x) \leq 1$ ae and $u_\varepsilon \to u_0$ in L^1, with $u_0 = \pm 1$ ae. Then

$$F_\varepsilon(u_\varepsilon) \geq 2 \int_\Omega |\nabla \varphi(u_\varepsilon)|$$

where $\varphi(t) = \int_{-1}^{t} |1-t^2|\,dt$ (we discussed this in Lecture

12). By dominated convergence, $\varphi(u_\varepsilon) \to \varphi(u_0)$ in L^1. By lsc of BV norm,

$$\liminf F_\varepsilon(u_\varepsilon) \geq 2\int_\Omega |\nabla\varphi(u_0)|$$

But

$$2\varphi(u_0) = \begin{cases} 0 & \text{where } u_0 = -1 \\ 8/3 & \text{where } u_0 = +1 \end{cases}$$

so

$$2\int |\nabla\varphi(u_0)| = \frac{8}{3}\, Per_\Omega \{u_0 = 1\}.$$

Sketch pf of ②: As explained in Lect 12, for equality to hold in

$$\int \varepsilon|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}|u_\varepsilon^2 - 1|^2 \geq 2\int |\nabla\varphi(u_\varepsilon)|$$

we would need $\varepsilon|\nabla u_\varepsilon| \simeq |u_\varepsilon^2 - 1|$. In 1D this is achieved by using $u_\varepsilon \simeq \pm\tanh\left(\frac{x - x_j}{\varepsilon}\right)$ near the j^{th} transition (modified far from x_j so that $u_\varepsilon = \pm 1$ exactly). In $\mathbb{R}^n$, $n \geq 2$, you get the same effect (as $\varepsilon \to 0$, with the geometry smooth enough + held fixed) by using

$$u_\varepsilon \approx \tanh\left(\frac{\text{dist}(x, A_0)}{\varepsilon}\right).$$

Since by pt C above it suffices to consider smooth sets A_K, this works.

(Making this arg't fully precise is a bit tedious.)

Pf of ⊖, ie cptness in L^1: recall that we have a unif bd on $\int |\nabla \varphi(u_\varepsilon)|$, where φ is monotone, $\varphi'(t) = |t^2-1|$.

Using the form of φ, we have $\varphi(t) \le C(1+|t|^3)$ so

$$\int_\Omega |\varphi(u_\varepsilon)| \le C \int_\Omega 1+|u_\varepsilon|^3$$

$$\le \text{const indep of } \varepsilon$$

using that $\frac{1}{\varepsilon}\int_\Omega |u_\varepsilon^2-1|^2$ stays bdd, and that $|u^2-1|^2 \sim u^4$ when $|u| \gg 1$.

So $\{\varphi(u_\varepsilon)\}$ stays bdd in BV, whence $\{\varphi(u_\varepsilon)\}$ stays cpt in L^1: any seq. has a subseq

$$V_{\varepsilon_j} = \varphi(u_{\varepsilon_j}) \to V_0 \quad \text{in } L^1$$

By unif cont'y of φ^{-1} we conclude that

u_{ε_j} converges in measure to $\varphi^{-1}(v_0)$.

Since u_ε are unif bdd in L^4, it follows that they converge to $u_0 = \varphi^{-1}(v_0)$ in L^1. (More detail:

$$\int_\Omega |u_{\varepsilon_j} - u_0| = \text{integral over } \Omega \cap \{|u_{\varepsilon_j} - u_0| \le \delta\}$$
$$+ \text{ integral over } \Omega \cap \{\delta < |u_{\varepsilon_j} - u_0| \le M\}$$
$$+ \text{ integral over } \Omega \cap \{|u_{\varepsilon_j} - u| \ge M\}.$$

1st term $\le \delta \cdot \mathrm{vol}(\Omega)$

2nd term $\to 0$ by convergence in measure, or by conv. of $\varphi(u_{\varepsilon_j})$ to $\varphi(u_0)$ in L^1,

3rd term $\le C M^{-3}$, since $|u_{\varepsilon_j} - u_0| \le M^{-3}|u_{\varepsilon_j} - u_0|^4$ on this set & we have unif bd in L^4.

Evidently $\limsup_{j\to\infty} \int |u_{\varepsilon_j} - u_0| \le C(\delta + M^{-3})$.

Letting $\delta \to 0$ & $M \to \infty$ gives the result.)

Now another example of Γ-convergence. Recall my 1st example of a pblm "in need of relaxation" (in Lecture 9) was the "elastic energy of a 2D sheet."

A very similar example involves the "elastic energy of a 1D string." If $\gamma : [0,L] \to \mathbb{R}^3$ represents the deformation of the midline, and W_{3D} is our 3D elastic energy, then the calc analogous to that of Lecture 9 says the "string" should be modeled by

$$\min \int_0^L W_{1D}(\gamma'(t))\, dt$$

where

$$W_{1D}(\xi) = \min_{\eta, \zeta \in \mathbb{R}^3} W_{3D}([\xi | \eta | \zeta])$$

3×3 matrix with cols ξ, η, ζ

But note that $W_{1D}(\xi)$ depends only on $|\xi|$ (due to frame-indifference of W_{3D}) and $W_{1D} \geq 0$ with $W_{1D}(\xi) = 0$ only for $|\xi| = 1$ (since W_{3D} prefers matrices in $SO(3)$). So W_{1D} is nonconvex! (If W_{3D} is incompressible neoHookean law then $W_{1D}(\xi) = |\xi|^2 + \frac{2}{|\xi|} - 3$, which is min at $|\xi| = 1$.)

What does Γ-conv have to say abt this?

Elastic String (E Acerbi, G Buttazzo, D Percivale, J Elasticity 25, 1991, 137-148)

Consider a (round) string with radius ε, and let

$$F_\varepsilon(u) = \frac{1}{\pi\varepsilon^2} \int_{\substack{0<x_1<L \\ x_2^2+x_3^2 \le \varepsilon^2}} W_{3D}(Du)\, dx_1\, dx_2\, dx_3$$

be the energy per unit area assoc with a 3D deformation u. Then F_ε Γ-converges to the 1D var'l pblm whose integrand is

$$\mathcal{C} W_{1D}(\gamma'),$$

ie the convexification (relaxation!) of our "1D elastic energy".

For the 2D elastic membrane a similar result holds; the main difference is that the Γ-limit is a 2D var'l pblm (involving maps $u: \mathbb{R}^2 \to \mathbb{R}^3$), so we must take the quasiconvexification $QW_{2D}(Du)$ which, depending on the form of W_{3D}, may or may not be the same as $\mathcal{C}W_{2D}$. (Surprisingly, for many examples including the incompressible

neoHookean case. it turns out that $QW_{2D} = \mathcal{C}W_{2D}$.
See Pipkin, IMA J Appl Math 36, 1986, 85-99.)
Membrane case was studied by H Le Dret + A Raoult,
J Math Pures Appl 74 (1995) 549-578.

Γ-convergence of elastic string example

① What topology to use? If we assume that $W_{3D}(Du) \sim C(1+|Du|^p)$ as $|Du| \to \infty$ then it is easy to see that cross-sectional averaging

$$\tilde{u}(x_1) = \frac{1}{\pi\varepsilon^2} \int_{x_2^2+x_3^2<\varepsilon^2} u(x_1, x_2, x_3)\, dx_2\, dx_3$$

has the property

$$F_\varepsilon(u) \text{ controls } \int_0^L |\partial_{x_1} \tilde{u}|^p\, dx_1$$

so it is natural to study Γ-conv wrto a topology that's cpt w.r.to unif bds in $W^{1,p}$. [For example, $L^1((0,L), \mathbb{R}^3)$ will do.]

② The "Γ-liminf ineq." We must show that if $\tilde{u}_\varepsilon \to u_0$ in L^1 then

$$\liminf F_\varepsilon(u_\varepsilon) \geq \int_0^L \mathcal{C}W_{1D}(u_0').$$

This is easy, since $\mathcal{C}W_{1D}$ (being convex) is lsc. In fact

$$F_\varepsilon(u_\varepsilon) = \frac{1}{\pi\varepsilon^2}\int_{\Sigma_\varepsilon} W_{3D}(Du_\varepsilon) \qquad \Sigma'_\varepsilon = \left\{ \begin{array}{l} 0 < x_1 < L \\ x_2^2 + x_3^2 < \varepsilon^2 \end{array} \right\}.$$

$$\geq \frac{1}{\pi\varepsilon^2}\int_{\Sigma'_\varepsilon} W_{1D}(\partial_{x_1} u_\varepsilon) \qquad \text{by defn of } W_{1D}$$

$$\geq \frac{1}{\pi\varepsilon^2}\int_{\Sigma'_\varepsilon} \mathcal{C}W_{1D}(\partial_{x_1} u_\varepsilon) \qquad \text{since } \mathcal{C}W_{1D} \leq W_{1D}$$

$$\geq \int_0^L \mathcal{C}W_{1D}(\partial_{x_1}\tilde{u}_\varepsilon)\, dx_1 \qquad \text{by Jensen, using convexity of } \mathcal{C}W_{1D}$$

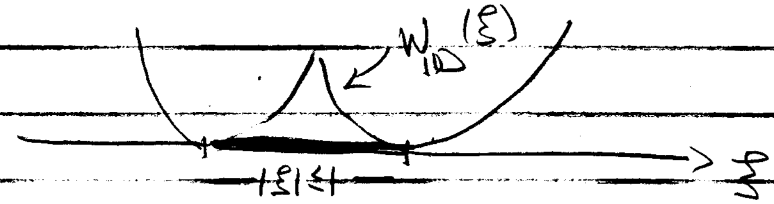
Now use lsc of last expression (which follows from convexity of $\mathcal{C}W_{1D}$).

(3) The "Γ-limsup ineq": we must show that given u_0, $\exists\, u_\varepsilon \to u_0$ st

$$F_\varepsilon(u_0) \approx \int_0^L \mathcal{C}W_{1D}(\partial_x u_0)\, dx$$

To keep matters simple, let's focus on the case (which is typical) that W_{1D} is convex when $|\partial_x u_0| > \underline{1}$, ie

$$\mathcal{C}W_{1D}(\xi) = \begin{cases} 0 & \text{if } |\xi| \le 1 \\ W_{1D}(\xi) & \text{if } |\xi| \ge 1 \end{cases}$$



Essential physics: compression can be avoided by folding instead.

Sketch of construction: given $u_0 : [0,L] \to \mathbb{R}^3$, approx it by a smooth curve. Then wherever $|\partial_x u_0| < 1$, introduce wrinkling to get a nearby unit-speed curve. Finally, put back the 3D deformation in 2 steps

step 1: defn of W_{1D} tells you what the dependence of Du_ε would "like to be" in the transverse vars x_2, x_3 if we ignore bending of the curve

step 2: accommodating bending leads to corrections, but this winds up being a "higher-order term" as $\varepsilon \to 0$.

Is this the "right" approach to an elastic string or membrane? Ans depends on your goal, or more precisely on the magnitude of the loads + bc.

If we want to consider situations where bending is dominant, then our normalization was wrong. For example, to consider shape of a piece of paper, held at 1 edge + bending due to force of gravity

3D pblm is $\int_{\Omega\times[0,h]} W_{3D}(Du)$. Ω = midplane of sheet

but normalization is h^3 (not h)

$$\frac{1}{h^3}\int_{\Omega\times[0,h]} W_{3D}(Du) \xrightarrow{\Gamma} \int_{\Omega} (\text{curvature})^2 \, dA$$

restricted to isometric immersions of Ω in $\mathbb{R}^3$

Essential physics: within a simple ansatz

$$\int_{\Omega\times[0,h]} W_{3D}(Du) \sim h\cdot\int_{\Omega} W_{2D}(\partial_{x_1}u, \partial_{x_2}u) + h^3\int_{\Omega}(\text{curvature})^2$$

If we normalize by h then curvature term is negligible as $h \to 0$ and we get relaxation of W_{2D}.

If we normalize by h^3 then we need $W_{2D}(\partial_{x_1}u, \partial_{x_2}u) \approx 0$, so W_{2D} forces isometry.

For more on this see G. Friesecke, R James, S Müller, "A hierarchy of plate models derived from nonlinear elasticity by Gamma-convergence", Arch Rat Mech Anal 180 (2006) 183-236