Calculus of Variations, Lecture 12, 4/23/2013

Fresh start today: the "Modica - Mortola problem", which (besides much intrinsic interest) provides a good introduction to Γ-convergence.

Some orientation first:

- We know that some var'l pbms require min sequences to be highly oscillatory, eg

$$\int_0^1 (u_x^2-1)^2 + \lambda u^2$$

and others permit osc behavior without requiring it, eg

$$\int_0^1 (u_x^2-1)^2$$

In either case, a higher-order term (and proper scaling) should select scale + local character of osc, eg

$$\int_0^1 (u_x^2-1)^2 + \varepsilon^2 u_{xx}^2 + \lambda u^2$$

or

$$\int_0^1 \frac{1}{\varepsilon}(u_x^2-1)^2 + \varepsilon u_{xx}^2 + \lambda u^2$$

- Var'l pbms with small parameters occur in lots of other settings, eg

3D elasticity in thin domains
$\rightarrow$ deformation of an elastic sheet

(one of our examples of a problem "in need of relaxation");

$$\int \frac{|u_i - u_{i+1}|^2}{(\Delta x)^2} \longrightarrow \int u_x^2$$

(a much more familiar + more <u>tame</u> convergence, since min sequences are not expected to be oscillatory)

Numerical analysis of var'l pbms is abt examples like the last one — design + analysis of schemes that avoid osc behavior, for well-posed var'l pbms assoc to limit $\Delta x \rightarrow 0$.

Γ-convergence is different: focus is on var'l pbms from physics or another appln where min sequences do have some osc or "gridscale" behavior. Goal is to understand the preferred small-scale behavior + identify a limiting var'l pbm that incorporates this understanding. (Note strong parallel to our descn of "relaxation.")

Modica - Mortola pbm is just one example, but a convenient one since it is both important + relatively easy. Some sources:

- Chapter 7 of Jost + Li-Jost isn't bad (it includes lots of technical pts I'll skip over) but note: specific version of Modica-Mortola pbm considered there is a bit different from my chosen focus

- Relatively easy reading, + basis of my descn abt "local minimizers" is RV Kohn + P Sternberg, "Local minimizers + singular perturbations", Proc Roy Soc Edinburgh 111A (1989) 69-84

- Early articles were: L. Modica + S. Mortola, Boll. Unione Mat. Italiana, 2 articles in Italian in 1977 (no vol constraint); then P. Sternberg, ARMA 101 (1988) 209-260 & L Modica, ARMA 98 (1987) 123-142 (with vol constraint).

- For broader perspective on Γ-convergence see A. Braides's book "Γ-convergence for beginners"

(OUP, avail thru Bobcat as an ebook, easy to read but limited to 1D problems) or A. Braides notes "A handbook of Γ-convergence" (avail at axp.mat.uniroma2.it/~braides/Handbook.pdf, publ in "Handbook of Diff'l Eqns: Stationary Partial Diff'l Eqns, Vol 3," Chipot + Quittner eds, Elsevier, 2006).

Getting started: our goal is to "understand asymptotic behavior of"

$$F_\varepsilon(u) = \int_\Omega \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon}(u^2-1)^2 \, dx$$

in limit as $\varepsilon \to 0$ (here $\Omega \subset \mathbb{R}^n$ is bounded + $u: \Omega \to \mathbb{R}$). Ans will be: F_ε "Γ-converges" to

$$F_0(u) = \begin{cases} \frac{8}{3} \operatorname{Per}_\Omega \{x : u(x) = 1\} & \text{if } u = \pm 1 \text{ ae} \\ \infty & \text{otherwise} \end{cases}$$

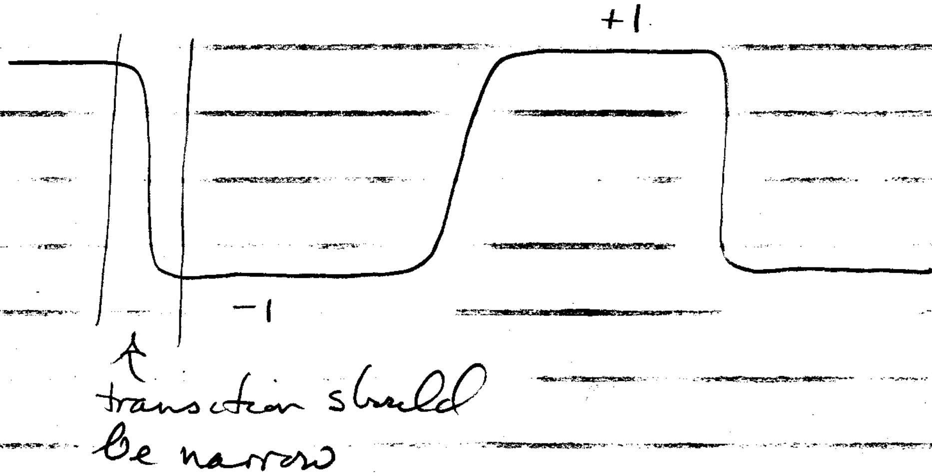
and as consequences:

a) if $G(u)$ is cpt under L^1 convergence, minimizers of $F_\varepsilon(u) + G(u)$ converge to minimizers of $F_0(u) + G(u)$

b) an isolated L^1-local min of $F_0(u)$ is the limit as $\varepsilon \to 0$ of L^1-local minimizers of F_ε.

(We'll explain these in due course; but note: the main point of Γ-convergence is to provide into about minimizers.)

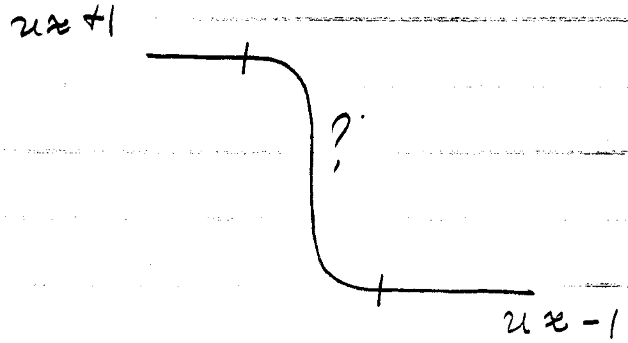
Some intuition first: suppose $\Omega = [0,1] \subset \mathbb{R}$ and ε is small. If F_ε is not of order $1/\varepsilon$ then $u^2 \approx 1$ so u should be close to ± 1 except on some "transitions"



1D Modica-Mortola asserts: as $\varepsilon \to 0$, energy required for a transition is exactly $8/3$.

It's easy to see why this should be true:

suppose $u \approx +1$ for $x \le a$ and $u \approx -1$ for $x \ge b$.



Then $$\int_a^b \varepsilon u_x^2 + \frac{1}{\varepsilon}(u^2-1)^2 \ge 2\int_a^b |u^2-1|\,|u_x|$$
$$= 2\int_a^b |\varphi(u)_x| = 2|\varphi(1)-\varphi(-1)|$$

where $\varphi'(t) = |t^2-1|$, eg $\varphi(t) = \int_{-1}^t |1-s^2|\,ds$, for which
$$-1 \le t \le 1 \Rightarrow \varphi(t) = t - \tfrac{1}{3}t^3 \Big|_{-1}^t \Rightarrow \varphi(1)-\varphi(-1) = 4/3,$$

This calcn also shows form of the optimal transition: it has

$$\sqrt{\varepsilon}\,|u_x| = \frac{1}{\sqrt{\varepsilon}}|u^2-1| \quad \text{ie} \quad \varepsilon u_x = \pm(1-u^2)$$

This is easily integrated; one finds that
$$u \simeq \pm \tanh\left(\frac{x-x_0}{\varepsilon}\right)$$
for an (optimal) transition centered at x_0.

Preceding arg't was purely variational - there was no PDE in sight - but it is related to method of matched asymptotic expansions. Our 1D fn'l

$$\int_0^1 \varepsilon u_x^2 + \frac{1}{\varepsilon}(u^2-1)^2$$

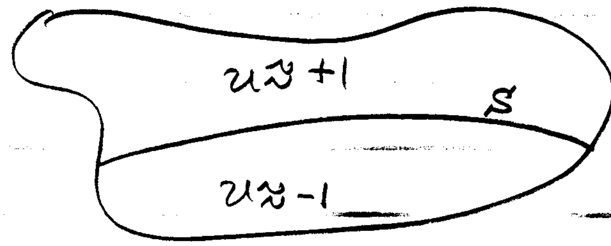
has only the obvious local minima $u \equiv \pm 1$ (this was an exercise earlier this semester) but it has plenty of saddle pts, which solve

$$-2\varepsilon u_{xx} + 4\varepsilon^{-1}(u^3-u) = 0 \quad \text{in } [0,1]$$
$$u_x = 0 \quad \text{at endpts}$$

Low-index saddles can be understood by phase plane analysis - the "transitions" are more or less evenly spaced. One could try to represent the soln of pde by matched asymptotic expansion. The "inner expansion" (describing a single transition btwn +1 & -1) would lead to same profile we derived above. (It's actually <u>difficult</u> to implement this fully, because the "outer expansion" involves only terms that are "exponentially small" as $\varepsilon \to 0$.)

Multidimensional picture is similar - let's

Focus on 2D for simplicity — but now "transitions" have more freedom (they're along curves, not at points). Still, assertion of Modica-Mortola is that to get



we require "energy" in the "transition layer near S" that's $8/3 \cdot \text{Length}(S)$.

Informal arg^t is as before:

$$\int_\Omega \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon}(u^2-1)^2 \geq 2\int_\Omega |\nabla u||u^2-1|$$

$$= 2\int_\Omega |\varphi'(u)\nabla u|$$

$$= 2\int_\Omega |\nabla \varphi(u)|$$

If $u_\varepsilon \to u_0$ (discontinuous, as in picture, $u_0 = \pm 1$ a.e.) then $\varphi(u_\varepsilon) \to \varphi(u_0)$ (takes values $\varphi(1)$ + $\varphi(-1)$, jumping across S) and

$$2\int |\nabla \varphi(u_0)| = 2[\varphi(1) - \varphi(-1)] \cdot \text{Length}(S)$$

since $\nabla \Phi(u_0)$ is "a δ-fn concentrated at S."

Moreover, arg't shows that ineq is (almost) sharp if $u_\varepsilon(x) = \pm \tanh\left(d(x,S)/\varepsilon\right)$, where $d(x,S)$ = signed distance to S.

More careful statements:

① if $v_\varepsilon \to v_0$ in L^1 as $\varepsilon \to 0$, then

$$\liminf_{\varepsilon \to 0} F_\varepsilon(v_\varepsilon) \geq F_0(v_0)$$

② if $v_0 \in L^1$ then $\exists\, v_\varepsilon \to v_0$ in L^1 s.t.

$$F_\varepsilon(v_\varepsilon) \to F_0(v_0) .$$

These constitute the <u>defn</u> of F_ε Γ-converging to F_0 in the L^1 topology. Also, we have

③ if $\{v_\varepsilon\}$ have $F_\varepsilon(v_\varepsilon)$ unif bdd, then $\{v_\varepsilon\}$ is cpt in $L^1(\Omega)$.

This is why it is natural, in this example, to study Γ-convergence wr to the L^1 topology.

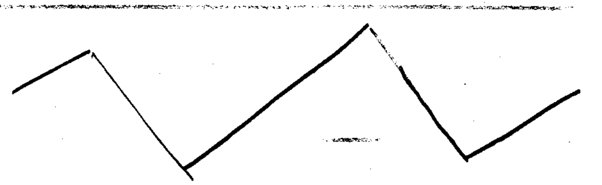
Our intuitive discn should have made ① – ③ plausible; we'll return to discuss them more carefully later. But now let's discuss their <u>consequences</u>

<u>1st consequence</u> (special case, in 1D, for simplicity): consider

$$E_\varepsilon(u) = \underbrace{\int_0^1 \varepsilon u_{xx}^2 + \frac{1}{\varepsilon}(u_x^2-1)^2}_{\text{Modica-Mortola wrt } u_x\text{ !}} + \lambda u^2$$

In limit $\varepsilon \to 0$, it Γ-converges to

$$E_0(u) = \int_0^1 \lambda u^2 dx + \frac{8}{3}\cdot \#(\text{teeth}), \quad \text{if } u_x = \pm 1$$
(a "sawtooth").



In particular, $\min_u E_\varepsilon \to \min E_0$, and if u_ε minimizes E_ε then limit pts of $\{u_\varepsilon\}$ are minimizers of E_0.

(Note: for E_0, the opt'l # of teeth depends on

λ; finding the optimal u for a given # teeth is a nonlocal var'l pblm, since constraint $u_x = \pm 1$ is very rigid.)

Since it's u_x (not u) that participates in the "Modica-Mortola terms" $\varepsilon u_{xx}^2 + \frac{1}{\varepsilon}(u_x^2-1)^2$, the relevant topology is $u_x \in L^1$ ie $W^{1,1}$ (though the limits are better: $W^{1,\infty}$). The lower order term λu^2 is cpt in this topology, so it does not affect the Γ-convergence (if $F_\varepsilon \xrightarrow[\Gamma]{} F_0$ as defined by ① + ② then $F_\varepsilon + G \xrightarrow[\Gamma]{} F_0 + G$ provided G is cont's on $W^{1,1}$).

<u>Claim</u>: Γ-convergence $\Rightarrow$ convergence of minimizers, and convergence of minimizing value.

<u>Pf</u>: if $F_\varepsilon \xrightarrow[\Gamma]{} F_0$ then

$$\limsup_{\varepsilon \to 0} (\min F_\varepsilon) \leq \min F_0 \quad \text{by ②}$$

but

$$\liminf_{\varepsilon \to 0} (\min F_\varepsilon) \geq \min F_0 \quad \text{by ①}$$

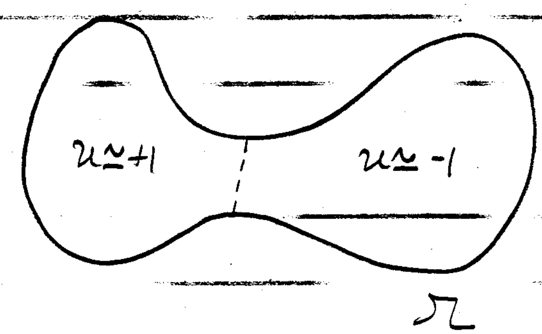
So minimizing values converge. So if u_ε minimizes F_ε, $F_\varepsilon(u_\varepsilon) \to \min F_0$. Now suppose $u_\varepsilon \to u_0$. Then

$$\liminf F_\varepsilon(u_\varepsilon) \geq F_0(u_0) \quad \text{by ①}$$

Thus $F_0(u_0) = \min F_0$; so u_0 achieves $\min(F_0)$.

(<u>Note</u>: preceding pf is general, ie not special to Modica-Mortola.)

<u>2nd consequence</u> (drawn from the Kohn-Sternberg paper, see pg 12.3 for full citation). Suppose Ω has a "strictly convex neck"



Then

A) The fnl $F_0 = \frac{8}{3} \operatorname{Per}_\Omega \{u=1\}$ if $u = \pm 1$

has an isolated L^1-local min u_0 assoc to minimal-length path crossing the neck

B) $F_\varepsilon = \int_\Omega \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon}(u^2-1)^2$ has a local min u_ε st $u_\varepsilon \to u_0$ as $\varepsilon \to 0$,

Pf of (A) is not trivial, but I won't do it here (see the paper).

Pf of ② is similar to what we did before: consider

$$\min_{\|u-u_0\|_{L^1} \le \delta} F_\varepsilon(u)$$

If ε is optimal, then either $\|u_\varepsilon - u_0\| = \delta$ or else it's a local min of F_ε in L^1 topology L^1.

<u>Claim</u>: For small ε we cannot have $\|u_\varepsilon - u_0\|_{L^1} = \delta$. In fact, if $\exists$ seq $\varepsilon_j \to 0$ st $\|u_{\varepsilon_j} - u_0\|_{L^1} = \delta$ then (using cptness) $\exists$ limit pt u_* st $\|u_* - u_0\|_{L^1} = \delta$, and part ① of Γ-conv defn says

$$\liminf_{\varepsilon_j \to 0} F_{\varepsilon_j}(u_{\varepsilon_j}) \ge F_0(u_*).$$

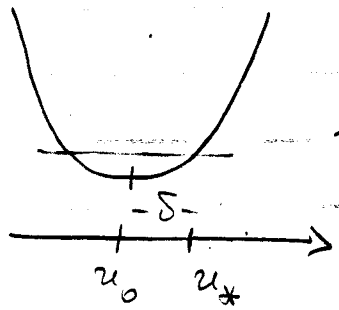
But pt ② of defn of Γ-conv tells us that

$$\limsup_{\varepsilon_j \to 0} F_{\varepsilon_j}(u_{\varepsilon_j}) \le F_0(u_0)$$

So

$$F_0(u_*) \le F_0(u_0).$$

But by (A), u_0 is an <u>L^1-isolated</u> local min of F_0, so if δ is small enough we get a contradiction (remembering that $\|u_* - u_0\|_{L^1} = \delta$).



schematic of F_0

(Claim is now proved.)

It's now clear that u_ε is an L^1-local min of F_ε for suff'tly small ε. Easy to see that $u_\varepsilon \to u_0$: otherwise $\exists$ seq ε_j s.t. $\|u_{\varepsilon_j} - u_0\|_{L^1} \geq c_0 > 0$. Then any limit pt u_* has $\|u_* - u_0\|_{L^1} \geq c_0$. But, as shown above, $F_0(u_*) \leq F_0(u_0)$, contradicting again hypoth that u_0 was an isolated local min.

It remains to say a bit more about how properties ①, ②, ③ are proved for the Modica-Mortola example. We'll address this (and if time permits consider another example) next week.

<u>Suggested exercises</u>:

(1) Recall our discussion why, in the 1D setting, if $u(a)=1$ and $u(b)=-1$ then

$$\int_a^b \varepsilon u_x^2 + \frac{1}{\varepsilon}(u^2-1)^2 \geq \frac{8}{3},$$

the inequality being very nearly sharp when u has a tanh profile (adjusted slightly near a + b to meet the bc)

Show that something similar can be done for

$$\int_a^b \varepsilon u_x^2 + \frac{1}{\varepsilon} W(u)\, dx$$

when W is smooth with

$W \geq 0$, and $W=0$ exactly at $u = \pm 1$.

(Note, however, that it makes a difference whether $W''(\pm 1)$ is strictly positive or not, since the way that $\int_0^a \frac{du}{W^{1/2}(u)} \to \infty$ as $a \to 1$ depends on this.)

(2) Now consider the analogue of pbm 1 where u is vector-valued, $u: [a,b] \to \mathbb{R}^n$, and the "potential" $W: \mathbb{R}^n \to \mathbb{R}$ prefers two points

in $\mathbb{R}^n$

$W \geq 0$, with $W = 0$ exactly at $\vec{u} = \xi$ & $u = \eta$

Show that

$$\min_{\substack{u(a)=\xi \\ u(b)=\eta}} \int_a^b \varepsilon u_x^2 + \frac{1}{\varepsilon} W(u) \geq d$$

where

d = "distance from ξ to η, in the metric on $\mathbb{R}^n$ with weight $W^{1/2}$"

$$= \min_{\substack{\vec{y}(0)=\xi \\ \vec{y}(1)=\eta}} \int_0^1 W^{1/2}(\vec{y}(t)) \, |\dot{y}(t)| \, dt.$$

(Hint: show that $W^{1/2}(u)\,|u_x| \geq |\partial_x \varphi(u)|$ for any $\vec{u} \in \mathbb{R}^n$, when

$\varphi(\vec{u})$ = "distance from ξ to $\vec{u}$, in the metric on $\mathbb{R}^n$ with weight $W^{1/2}$"

$$= \min_{\substack{\vec{y}(0)=\xi \\ \vec{y}(1)=\vec{u}}} \int_0^1 W^{1/2}(\vec{y}(t)) \, |\dot{y}(t)| \, dt \quad .).$$