Calculus of Variations, Lecture 10, 4/9/2013

Motivated by the examples from Lecture 9, we now ask: can we evaluate

$$\min_{bc} \int_\Omega W(Du)$$

when $W(Du)$ is not lower-semicontinuous (ie when it is "in need of relaxation")?

A closely-related question is: for $u: \mathbb{R}^n \to \mathbb{R}^m$, $n \geq 2$, $m \geq 2$, how can we tell whether $\int W(Du)$ is lower-semicontinuous or not?

Major sources for what follows:

a) Section 2 (only) of my article with M Vogelius, "Relaxation of a variational method for impedance computed tomography," CPAM 40 (1987) 745-777

b) Dacorogna's book

My focus is on pbms of form $\int W(Du)\,dx$ for simplicity only — no new ideas are needed to handle lower-order terms, $\int W(x,u,Du)\,dx$, etc. I'll assume throughout that W grows like $|Du|^p$

at ∞ (so a bd on $\int N(Du)$ implies a bound on u in $W^{1,p}(\Omega)$).

Discn of relaxation will center on

$$QW(F) = \inf_{\varphi|_{\partial U} = F\cdot x} \frac{1}{|U|} \int_U W(D\varphi)\,dx$$

$=_{\text{dfn}}$ "quasiconvexification" of W, also called "relaxation" of W.

Conceptually: if we view $W(Du)$ as an "energy density," then

$QW(F) =$ min (energy/unit vol), if average gradient is F

Facts: ① value obtained for QW is indep of choice of domain U; also there's an equiv defn with periodic bc

$$QW(F) = \inf_{\psi \text{ periodic}} \int_{[0,1]^n} W(F+D\psi)\,dx$$

② when $u: \mathbb{R} \to \mathbb{R}^n$ or $u: \mathbb{R}^n \to \mathbb{R}$, QW is simply the convexification of W

(ie the largest convex integrand $\leq W$)

We'll prove these assertions later. First let's explain the importance of QW

<u>Numerically oriented discussion</u>:

a). We could attempt to minimize $\int W(Du)$ via finite element discretization, using piecewise linear fns on a particular triangulation. (If W is nonconvex this var'l pbm could be numerically intractable, with lots of local minima.)

b). We'll get a smaller value if we enlarge space of test fns to be those whose restriction to the <u>skeleton</u> of the triangulation is piecewise linear (ie we permit arbitrary variation within each triangle).

- Procedure (b) is <u>equivalent</u> to minimizing $\int_\Omega QW(Du)$ using piecewise-linear elements on the given triangulation.

- This is clearly practical only if we can determine QW analytically

- "relaxed problem" $\int_\Omega QW(Du)$ is often better-behaved numerically than $\int_\Omega W(Du)$ [For example: if $u: \mathbb{R} \to \mathbb{R}^n$ or $\mathbb{R}^n \to \mathbb{R}$, relaxed pbm is convex. However relaxation typically produces degeneracy; for example in our 2nd example of Lecture 9, $\int |\phi|$ is convex but not strictly convex.]

<u>Analytically-oriented discussion</u>:

a) relaxed pbm has same min as original one (clear from preceding discn)

b) relaxed pbm is lower semicontinuous, so it <u>has</u> a soln (perhaps more than one!)

c) from any minimizer of relaxed pbm we get a recipe for constructing a minimizing sequence for the original pbm [clear from numerical discn]

d) wk limit of any min sequence for orig pbm must minimize the relaxed pbm

[Easy pf: if $\int W(Du^n)dx \downarrow \min$, then $\int QW(Du^n)\,dx \downarrow \min$ by (a) combined with $QW \le W$; assertion is clear now from (b), i.e., from lower semicontinuity of QW.]

(e) $\int_\Omega W(Du)\,dx$ is lsc iff $QW = W$.

Most of preceding assertions are already clear. The ones that require pf are

(i) assertion that QW is indep of domain (and equiv of the defn using periodic bc)

(ii) QW is quasiconvex (ie $Q(QW) = QW$)

(iii) $\int_\Omega f(Du)\,dx$ is lsc iff f is quasiconvex

(Note: taken together, (ii) + (iii) prove (b) + (e) of the "analytically-oriented discussion".)

When we are done with (i)–(iii), there will remain only the (crucial) question how to <u>find</u> QW, for a given W.

Aboref (i) : Let $Q_1 W$ + $Q_2 W$ be defined as in pg 10.2, but using two different domains U_1 + U_2. Fixing F, choose φ_1 st $\varphi_1 \big|_{\partial U_1} = F \cdot x$ and

$$\frac{1}{|U_1|} \int_{U_1} W(D\varphi_1)\, dx \le Q_1 W(F) + \varepsilon .$$

We can scale φ_1 to live on any translated + scaled copy of U_1, say

$$U_1' = \{ x : x - x_0 \in \lambda U_1 \}$$

eg if $U_1 = B_1(0)$, U_1' could be $B_\lambda(x_0)$ with

$$\varphi_1'(x) = \lambda \varphi_1 \left(\frac{x - x_0}{\lambda} \right) + F \cdot x_0$$

Note that

$$\frac{1}{|U_1'|} \int_{U_1'} W(D\varphi_1')\, dx = \frac{1}{|U_1|} \int_{U_1} W(D\varphi_1)\, dx,$$

and $\varphi_1'(x) = F \cdot x$ at $\partial U_1'$.

Now: pack U_2 (ae) by copies of U_1 (suitably scaled). Conclude that resulting test fn $\tilde{\varphi}$ has

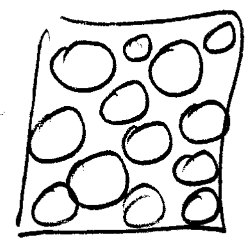
$$\int_{U_2} W(D\tilde{\varphi}) = \sum_j \int_{\substack{\text{scaled +} \\ \text{translated } U_1}} W(D\varphi_1').$$

$$\leq \sum' |U_1^{(j)}| (Q_1 W(F) + \varepsilon)$$

$$= |U_2| (Q_1 W(F) + \varepsilon)$$

As $\varepsilon \to 0$ get $Q_2 W(F) \leq Q_1 W(F)$. By symmetry, opposite holds as well. So $Q_1 W(F) = Q_2 W(F)$.

[Note: I didn't really need to cover U_2 ae by scaled copies of U_1; covering all but ε of it is enough to make the arg't work.]

Visual aid for preceding arg't: if eg U_2 = square, U_1 = circle  .

Still part of (i): alternative charzn of QW using periodic bc: I claim

$$QW(F) = \inf_{\substack{D\varphi \text{ periodic} \\ ⨍ D\varphi = F}} \int_{[0,1]^n} W(D\varphi) \, .$$

In fact: call this expression $Q_{per} W(F)$. Obviously $Q_{per} W \leq QW(F)$ (this is clearest if we write $\varphi(x) = u(x) + F \cdot x$, so that

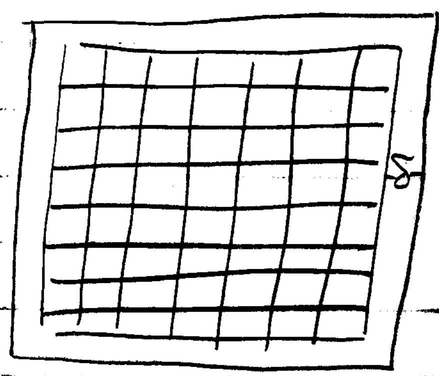
$$Q_{per}W(F) = \inf_{u \text{ periodic}} \int_{[0,1]^n} W(F+Du)\,dx;$$

since $u|_{\partial[0,1]^n} = 0$ is stronger than periodicity, every test fn for QW can also be used for $Q_{per}W$).

Sketch of converse: Choose u periodic st

$$\int_{[0,1]^n} W(F+Du) \le Q_{per}W(F) + \varepsilon$$

and construct φ as indicated:



- bdry layer thickness δ
- at outer bdry, $\varphi = F \cdot x$
- in interior, $\varphi = F \cdot x + \frac{1}{N} u(Nx)$

<u>Claim</u>: if N is suff'tly large (and if u is Lipschitz cont's) then bdry layer can be filled in in such a way that

$$|D\varphi|_{L^\infty} \le \text{const indep of } \delta$$

(Justifn uses Kirzbraun's Theorem, which says that if a fn defined on part of $\mathbb{R}^n$ has Lip const K <u>where defined</u>, then $\exists$ extnsn to $\mathbb{R}^n$ with same Lip constant.)

Accepting the claim:

$$\int_{[0,1]^n} W(D\varphi) = \int_{[0,1]^n} W(F+Du) + \mathcal{O}(\delta)$$

As $\delta \to 0$ this shows $QW(F) \leq Q_{per}W(F)$.

Proof of (ii): claim is that QW is quasiconvex, ie $Q(QW) = QW$.

Suppose not; consider if $\exists\ \varphi$ st $\varphi\big|_{\partial U} = F\cdot x$ and

$$\frac{1}{|U|}\int_U QW(D\varphi)\,dx < QW(F).$$

May suppose U is a polygon and u is piecewise linear on some triangulation of U (by a standard approxn thm). Recalling our "numerically-oriented discussion", we easily derive from φ a new test fn $\tilde{\varphi}$ st $\tilde{\varphi} = F\cdot x$ on ∂U and

$$\frac{1}{|U|}\int_U W(D\tilde{\varphi}) < QW(F).$$

But this contradicts the defn of QW.

About (iii): equivalence of lsc + quasiconvexity

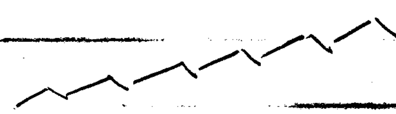
Half this assertion is easy: if $QW < W$ ("W is not quasiconvex") then $\int W(D\varphi)\,dx$ is not lsc under weak convergence. For example, use periodic chargn: suppose $D\varphi$ is periodic + $⨍ D\varphi = F$ and $⨍_{[0,1]^n} W(D\varphi) < W(F)$. Then

$$\varphi_N(x) = \frac{1}{N}\varphi(Nx)$$

has $⨍_{[0,1]^n} D\varphi_N = F$ + $\varphi_N \xrightarrow{\text{wkly}} \underbrace{F\cdot x + \text{const}}_{\varphi_\infty}$

But $\int_{[0,1]^n} W(D\varphi_\infty) = W(F) > \lim_{N\to\infty} \int_{[0,1]^n} W(D\varphi_N)$.

(This arg't is just the multi-dim'l analogue of our familiar picture that

$$u_x = \pm 1 \Rightarrow ax + \frac{1}{N}u(Nx)$$ 

converges wkly to $a\cdot x$)

The other half of the assertion is more subtle

It says that if $QW = W$ ("W is quasiconvex") then $\int QW(D\varphi)$ is lsc. For full proof see Dacorogna's book (in 1st edn it is pp 158-166).

Main ideas:

(a) if $\varphi_j \to \varphi_\infty$ wkly $W^{1,p}$ and u_∞ is affine and $\varphi_j|_{\partial\Omega}$ has same affine bc, then assertion is trivial (by defn of quasiconvexity).

(b) if φ_∞ is affine but we have no bc for φ_j, we can still say $\varphi_j|_{\partial\Omega} \to$ affine map (in suitable norm), since bdry trace is compact (eg as map $W^{1,p}$ to $L^p(\partial\Omega)$). When j is large, we can use this to modify φ_j near the bdry to make it exactly affine at $\partial\Omega$, without changing the "energy" much.

(c) any wk limit $\varphi_\infty \in W^{1,p}$ can be approx by a piecewise affine map, with almost the same "energy". Now see (b).

Suggested exercises:

(1) Show that $\inf_{u = F\cdot x \text{ at } \partial\Omega} \int_\Omega W(Du) + |u - F\cdot x|^2 \, dx$

achieves its minimum if and only if W is quasiconvex.

(2) Show that a quadratic form

$$W(Du) = \sum a_{i\alpha j\beta} D_\alpha u^i D_\beta u^j$$

is quasiconvex if and only if

$$\sum a_{i\alpha j\beta} \xi_i \xi_j \eta_\alpha \eta_\beta \geq 0$$

for all $\xi \in \mathbb{R}^n$ + $\eta \in \mathbb{R}^m$ (if $u: \mathbb{R}^m \to \mathbb{R}^n$).
[Hint: use the characterization involving periodic bc, and Plancherel's Theorem.]