Calculus of Variations - Lecture 9 - 11/11/09

Today: continue study of lowersemicont'y + relaxation (quasiconvexification), via

a) nec or sufft conds for lsc'y
b) upper or lower bds for QW
c) a few examples

Recall main pts from last time:

1) $QW(F) = \min_{u = F \cdot x} \frac{1}{|\Omega|} \int_\Omega W(Du)$

2) min sequence of "orig" var'l plm converge weakly to minimizers of "relaxed" plm

3) relaxed plm $\min_{bc} \int QW(Du)$ is lsc; its minimizer provides recipe for constr. min seq. of original plm.

4) W is lsc $\iff$ $QW = W$.

What can we say abt QW?

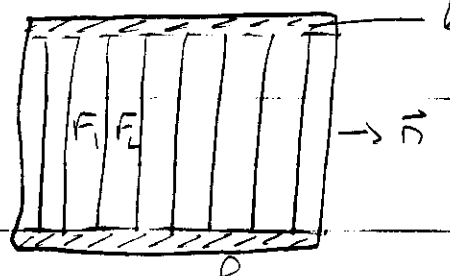
Layering lemma: Suppose $F = \theta F_1 + (1-\theta) F_2$ with $0<\theta<1$ + $F_2 - F_1 = a \otimes n$ (rank one). Then $\exists\, u^h$ s.t.

$Du^h = F_1$ or F_2 except on a set of measure $\to 0$ (with vol fraction $\approx \theta + (1-\theta)$ resp)

$|Du^h|$ unif bdd (indep of h)

$u^h = F \cdot x$ at bdry

Proof: use layering with normal $\vec{n}$ + length scale $h \to 0$, + a bdry layer with thickness $\sim h$



(Details left as exercise). Note that if u is scalar-valued then $F_1 + F_2$ are vectors (so the cond that $F_2 - F_1$ be rank one is always satisfied).

Consequence of layering lemma:

a) $QW(\theta F_1 + (1-\theta)F_2) \leq \theta W(F_1) + (1-\theta) W(F_2)$ if $F_2 - F_1$ is rank-one

[obvious]

b) $QW(\theta F_1 + (1-\theta)F_2) \leq \theta\, QW(F_1) + (1-\theta)\, QW(F_2)$ if $F_2 - F_1$ is rank-one [replace the piecewise-linear test fn of layering lemma with a more oscillatory one, using defn of $QW(F_1)$ + $QW(F_2)$].

c) for $u: \mathbb{R}^n \to \mathbb{R}$, QW = largest convex fn $\leq W$
= "convexn of W"

infact, (b) shows that QW is convex; but $u = F\cdot x \Rightarrow \frac{1}{|\Omega|}\int_\Omega Du = F$, so Jensen's ineq tells us that if φ is convex + $\varphi \leq W$ then

$$QW(F) = \inf_{u=F\cdot x} \frac{1}{|\Omega|}\int_\Omega W(Du)$$

$$\geq \inf_{u=F\cdot x} \frac{1}{|\Omega|}\int \varphi(Du)$$

$$= \varphi(F)$$

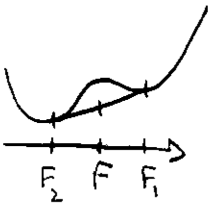
ie φ convex + $\varphi \leq W \Rightarrow \varphi \leq QW$. QED

More constructive version: let CW = convexn of W. Then for any F, must show $\exists u$ st

$$u\Big|_{\text{bdry}} = F\cdot x$$

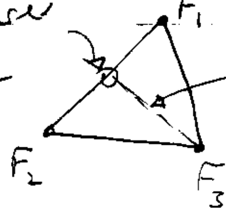
$$\frac{1}{|\Omega|}\int_\Omega W(Du) \approx CW(F).$$

Note: supergraph of CW is convex set in $\mathbb{R}^{n+1} \Rightarrow$ any pt is convex hull of (finitely many) extreme pts.

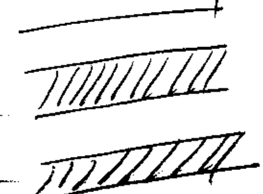


just two extreme pts $\Rightarrow$ use layering lemma once;
3 extreme pts $\Rightarrow$ use layering lemma twice

mixture of F_1+F_2

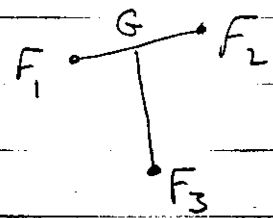


"2nd rank laminates"
mix F_3 with mixture of F_1+F_2



d) for $u: \mathbb{R}^2 \to \mathbb{R}^2$ we can still use 2nd (or higher) rank laminates to get upper bds on QW. For example:

if



F_2-F_1 is rank one
$G = \theta F_1 + (1-\theta) F_2$
$F_3 - G$ is rank one

Then
$$QW\left(\psi F_3 + (1-\psi)\theta F_1 + (1-\psi)(1-\theta) F_2\right) \le \psi W(F_3) + (1-\psi)\theta W(F_1) + (1-\psi)(1-\theta) F_2 .$$

What about lower bds on QW? Observing that

$$QW(F) = \min \int W(\lambda)\, d\mu(\lambda)$$

μ = distribution measure of ∇u for u st $u|_{\partial\Omega} = F \cdot x$

we see that looking for lower bds $\Leftrightarrow$ looking for possible restrictions on such distr measures μ. In scalar-valued setting there are none (other than trivial restrn $\int \lambda \, d\mu(\lambda) = F$) but in vector-valued case we get restrns from wk cont'y of null-Lagrangians.

Focus on $u: \mathbb{R}^2 \to \mathbb{R}^2$. Then

a) $QW \geq$ [largest polyconvex fn $\leq W$]

b) $$QW(F) \geq \min_{\substack{\int \lambda \, d\mu = F \\ \int \det(\lambda)\, d\mu = \det F}} \int W(\lambda) \, d\mu(\lambda)$$

where μ ranges over prob meas on 2x2 matrices.

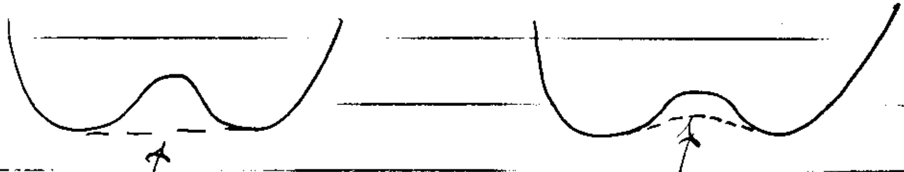
Actually, these two lower bds are <u>the</u> <u>same</u>

Direct pf of (a): argue essentially as we did in scalar case, to show that $\varphi \leq W$ + φ convex $\Rightarrow \varphi \leq QW$. (The only difference: now let φ be polyconvex.)

Direct proof of (b): obvious, since if μ is distrn measure assoc to u + $u|_{\partial\Omega} = F \cdot x$ then $\int \det(\lambda) \, d\mu = \frac{1}{|\Omega|} \int_\Omega \det Du = \det F$ by crucial property of det.

(Equiv of (a) + (b) was discussed in a CPAM paper by N. Firoozye.)

Example: Suppose A + B are 2×2 matrices with rank $(B-A)=2$. Suppose $W \geq 0$ + $W=0$ at $A+B$ only.
Claim: $QW=0$ only at $A+B$



convexn quasiconvexn.

Use charzn (b) (note that the RHS is lsc, so an optimal μ exists).

Sufft to consider $F = tA+(1-t)B$ (since otherwise $CW>0$, + $W \geq CW$). Must show that if μ is admissible for (b) then it cannot be sptd only at $A+B$ (if $t\neq 0$, $t\neq 1$). If it were, then $\mu = t\delta_A + (1-t)\delta_B$
$\Rightarrow$

$$\det(tA+(1-t)B) = \int \det\lambda \, d\mu = t\det A + (1-t)\det B$$

To see this is impossible, it's convenient to take $A=0$ + $B=I$ (no loss of generality – that's achieved by a lin change of vars). Then

$$\det(tA+(1-t)B) = (1-t)^2$$
$$t\det A + (1-t)\det B = 1-t$$

so $(1-t)^2 = 1-t$ i.e. $t=0$ or $t=1$.

Example 2: consider our 2D version of "two martensite wells": suppose $W(Du) \geq 0$ with $W=0$ exactly on $SO(2)U_1$ & $SO(2)U_2$ with

$$U_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \qquad U_2 = \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}$$

Where is $QW=0$? (Note: our earlier formulation with $W=0$ on $SO(2)\begin{pmatrix} 1 & \pm\delta \\ 0 & 1 \end{pmatrix}$ is equivalent to the one just given, in a rotated coord. system, if $\alpha\beta = 1$.)

Answer: $QW(F) = 0 \iff$ ① $\det F = \alpha\beta$

② $F^T F = \begin{pmatrix} D_{11} & D_{12} \\ D_{12} & D_{22} \end{pmatrix}$ satisfies

$$D_{11} + D_{22} + 2D_{12} \leq \alpha^2 + \beta^2$$

$$D_{11} + D_{22} - 2D_{12} \leq \alpha^2 + \beta^2$$

Proof that $QW(F) = 0 \Rightarrow$ ① & ②: If $QW(F)=0$ then $\exists$ distr meas μ st $\int \lambda \, d\mu = F$ & μ is sptd on $SO(2)U_1 \cup SO(2)U_2$ (and $\int \det \lambda \, d\mu = \det F$).

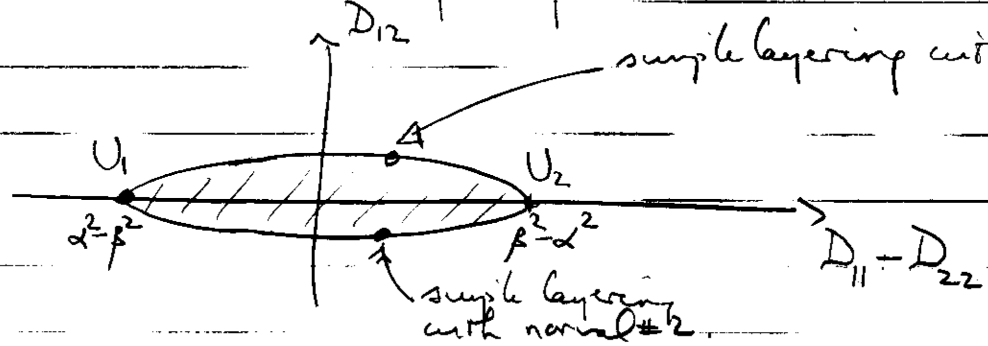
property ① follows immediately, since $\det \lambda \equiv \alpha\beta$ on $SO(2)U_1$ & $SO(2)U_2$.

property ② is elementary: for any vector $\vec{e}$,

$$|Fe|^2 = \langle F^T F e, e \rangle \le \max_{i=1,2} |U_i e|^2$$

Apply this to $e=(1,1)$ + $e=(1,-1)$ to get the two parts of (2).

Sketch of pf that ① + ② ⇒ $QW(F)=0$: successful constrn uses "rank-2 layering".



(space of possible D_{ij} is 2D surface in $\mathbb{R}^3$ due to constraint $\det D = \alpha^2\beta^2$; this figure is its proj'n)

Magic that makes constrn work:
- upper + lower bdries are achievable by rank 1 layering (with the two possible normals)
- corresp pts on upper + lower bdries corresp to F's that are rank-one related!

(For details see eg Bhattacharya's book on shape memory materials.) [Why did this work? No idea: in general, there's no assurance that the "polyconvexity" will be equal to the relaxation.]

Suggested problems:

① Show that

$$\min_{\int \lambda \, d\mu = F} \int W(\lambda)\, d\mu(\lambda)$$

is the largest convex function $\leq W$ (the "convexification of W," $\mathcal{C}W$). Here μ ranges over all probability measures.

② Here is another method for bounding QW from below:

$$QW \geq \mathcal{C}(W-g) + g$$

whenever g is quasiconvex. Prove it.

③ Let a_1 + a_2 be a pair of 2x2 matrices, + consider the "two quadratic wells" energy

$$W(F) = \min\left\{ \tfrac{1}{2}|F-a_1|^2,\ \tfrac{1}{2}|F-a_2|^2 \right\}$$

where F is 2×2. Show that

(*) $$QW(F) \geq \min_{0\leq\theta\leq 1} \tfrac{1}{2}|F-\bar{a}(\theta)|^2 + \frac{\theta(1-\theta)}{2} h.$$

where
$$\bar{a}(\theta) = \theta a_1 + (1-\theta) a_2$$
and
$$h = |a_1 - a_2|^2 - \max_{|k|=1} |(a_1-a_2)k|^2$$

by applying problem (2) with
$$g(F) = \tfrac{1}{2}|F|^2 - c\langle F, a_2 - a_1\rangle^2$$

and c approaching
$$c_0 = \tfrac{1}{2}\left(\max_{|k|=1} |(a_2-a_1)k|^2\right)^{-1}$$

(4) Show that relation (*) holds with equality, ie
$$QW(F) = \min_{0\leq\theta\leq 1} \tfrac{1}{2}|F-\bar{a}(\theta)|^2 + \frac{\theta(1-\theta)}{2} h$$

[Hint: an upper bound requires a construction. In this case a suitable application of the layering lemma suffices.]