Calculus of Variations - Lecture 8 - 11/4/09

Topic: relaxation of nonconvex var'l problems.

Sources + places to read more:

- chapter 5 of Jost + Li-Jost has some of what we'll do, but personally I find the treatment compressed + lacking in examples
- Dacorogna's book includes almost everything we'll do (for a viewpoint that emphasizes the role of "Young measures" see also books by Pablo Pedregal)
- for a fast, nontechnical summary of basic facts abt the "quasiconvexification" see Section 2 (only) of article "Relaxation of a variational method for impedance computed tomography" (RV Kohn + M Vogelius) CPAM 40 (1987) 745-777 (available online @ CPAM website)

Overall goal: for problems of form

$$(*) \qquad \min_{u|_{\partial\Omega} = u_0} \int_\Omega W(Du)\, dx$$

understand a) nec/suff't conds for lsc'y (so existence can be proved by direct method)

b) systematic approach to finding min sequences + value of min energy (when fnl is not lsc)

Focus on $\int W(Du)\,dx$ is for simplicity only — no new ideas are required to handle lower-order terms, $\int W(x,u,Du)\,dx$, etc. I'll assume throughout that W grows like $|Du|^p$ near ∞ (so bd on "energy" $\Rightarrow$ bd on u in $W^{1,p}(\Omega)$).

Object of primary interest is "quasiconvexification of W"

$$QW(F) = \inf_{\varphi|_{\partial U} = F\cdot x} \frac{1}{|U|}\int_U W(D\varphi)\,dx$$

Conceptually,

$$QW(F) = \min\ (\text{energy/unit vol}),\ \text{if average gradient is } F.$$

<u>Fact</u>: value obtained for QW is indep of choice of domain U; also there's an equiv defn using periodic bc

$$QW(F) = \inf_{\psi \text{ periodic}} \int_{[0,1]^n} W(F+D\psi)$$

(We'll return to proof in a moment.)

Numerically oriented expln for importance of QW:

a) we could attempt to minimize $\int W(Du)$ via finite element method, using (say) p.l. cont's fns on a particular triangulation. (If W is nonconvex, this var'l pbm could have lots of local minima + be numerically intractable)

b) we can get a smaller value by enlarging space of test fns to consist of those whose restrictions to skeleton of triangulation are pl. (ie we permit arbitrary variation within each triangle).

- procedure (b) is equivalent to solving $\min_{u=u_0 \text{ on } \partial\Omega} \int_\Omega QW(Du)$ on the given triangulation
- obviously practical only if we can say something abt QW analytically
- "relaxed pbm" $\int_\Omega QW(Du)$ is often better-behaved numerically than original (if $u: \Omega \to \mathbb{R}$ or $u: \mathbb{R} \to \mathbb{R}^n$

it's convex – we'll show this later; otherwise it's not nec. convex, but still it seems there are not many local optima in most cases. However relaxation often produces degeneracy – eg example $\int |7u|$ in Lecture 7 is convex but not strictly convex.)

Analytically-oriented expln for importance of QW:

a) relaxed problem has same min value as original one [clear from preceding discn]

b) relaxed plm is lsc, so it has a soln (perhaps more than one!)

c) from any minimizer of relaxed plm we get a recipe for constructing min sequences of orig plm [clear from numerical discn]

d) wk limit of a min seq. for orig plm must be a minimizer of relaxed plm, since $\int W(Du^n)\,dx \downarrow \min \Rightarrow \int QW(Du^n) \downarrow \min$ [using (a), and obvious reln $QW \le W$]; assertion follows now, from lsc of QW [assertion (b)].

e) $\int_\Omega W(Du)$ is lsc iff $QW = W$

("W is quasiconvex").

We'll prove (or sketch proofs of) these results. Then we'll be left with question: can we evaluate QW analytically?

1^{st} pf: independence of defn of QW wrt choice of domain U.

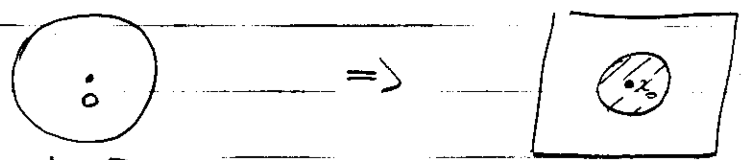
Let $Q_1 W$ + $Q_2 W$ be defined using domains U_1 + U_2 resp (in defn on pg 8.2). Fixing F, choose φ_1 st $\varphi_1|_{\partial U_1} = F \cdot x$

$$\frac{1}{|U_1|} \int_{U_1} W(D\varphi_1)\, dx \leq Q_1 W(F) + \varepsilon$$

We can scale φ_1 to live on any translated + scaled copy of U_1, say

$$U_1' = \{ x : x - x_0 \in \lambda U_1 \}$$

eg



$U_1 = B_1(0)$ $\Rightarrow$ $U_1' = B_\lambda(x_0)$

U_2 = cube, perhaps

$\varphi_1'(x) = \lambda \varphi_1\left(\frac{x - x_0}{\lambda}\right) + F \cdot x_0$

Note ~~that~~ $\frac{1}{|U_1'|}\int_{U_1'} W(D\varphi_1') = \frac{1}{|U_1|}\int_{U_1} W(D\varphi_1)$

and $\varphi_1'(x) = F\cdot x$ at $\partial U_1'$.

Now we can pack U_2 (ae) by copies of U_1 (suitably scaled) using this constr on each copy. Conclude that for the resulting test fn $\tilde{\varphi}$,

$$\int_{U_2} W(D\tilde{\varphi}) = \sum_j \int_{\substack{\text{translated} \\ \text{+ scaled copy} \\ \text{of } U_1}} W(D\varphi_1').$$

$$\leq \sum |U_1^{(j)}| \left(Q_1 W(F) + \varepsilon\right)$$

$$= |U_2| \cdot \left(Q_1 W(F) + \varepsilon\right)$$

As $\varepsilon \to 0$ conclude $Q_2 W(F) \leq Q_1 W(F)$. By symmetry $Q_2 W(F) = Q_1 W(F)$. <u>QED</u>

Alternative charzn using periodic test fns:

$$QW(F) = \inf_{\psi \text{ periodic}} \int_{[0,1]^n} W(F + D\psi)$$

$$= \inf_{\substack{D\varphi \text{ periodic} \\ \int D\varphi = F}} \int_{[0,1]^n} W(D\varphi).$$

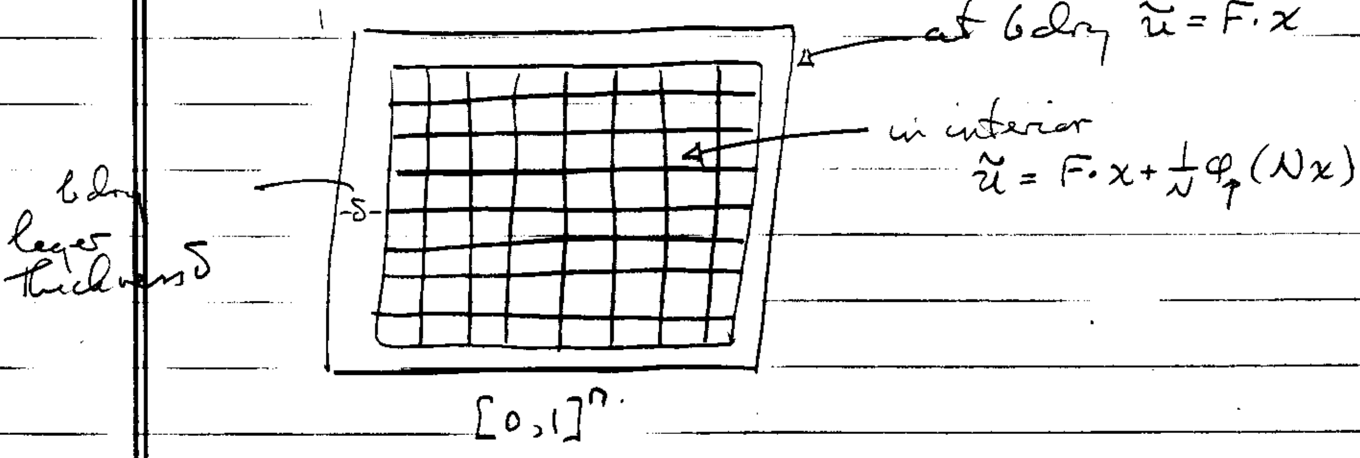
Pf: Call this inf $Q_p W(F)$. Obviously

$$Q_p W(F) \leq QW(F)$$

since if $u = F \cdot x$ at $\partial\Omega$ then $\varphi(x) = u(x) - F\cdot x$ qualifies as a test fn for $Q_p W$. For converse, choose φ_p periodic st

$$\int_{[0,1]^n} W(F + D\varphi_p) \leq Q_p W(F) + \varepsilon$$

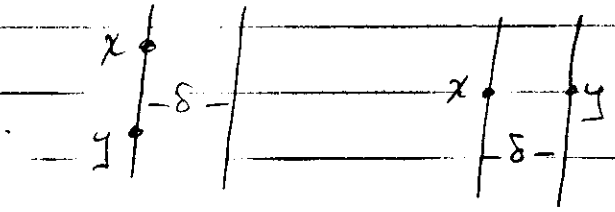
+ construct assoc $\tilde{u}$ as indicated:



Claim: if N is suffly large then $\tilde{u}$ can be filled in on bdry layer in such way that

$$|D\tilde{u}|_{L^\infty} \leq \text{const indep of } \delta$$

Pf uses Kirzbraun's Thm, which says it suffices to check Lipschitz const of $\tilde{u}$ on the set where it has already been defined.



$|\tilde{u}(x) - \tilde{u}(y)| \le (|D\varphi_p|_\infty + |F|)\, |x-y|$ for LHS picture using Fund Thm Calc

$|\tilde{u}(x) - \tilde{u}(y)| \le |F|\,|x-y| + \frac{1}{N} |\varphi_p|_\infty \le 2|F|\,|x-y|$ if $N \gg 1$ for RHS picture (good choice of N depends on δ)

Evidently, $\tilde{u}$ has affine bc and

$$\int_{[0,1]^n} W(D\tilde{u}) = \int_{[0,1]^n} W(F + D\varphi_p) + O(\delta)$$

As $\delta \to 0$ this shows $QW(F) \le Q_p W(F)$.

Assertions (a), (b) of "numerically oriented discn" are elementary (except for last bullet - we'll discuss character of QW as integrand later).

Assertions a, c, d of "analytically oriented discn" are elementary.

To justify (b) + (e), we'll sketch pfs that

- QW is quasiconvex (ie $Q(QW) = QW$)
- $\int f(Du)$ is lsc iff f is quasiconvex

For 1st bullet: suppose $\exists\, u$ st $u|_{\partial U} = F \cdot x$ and

$$\frac{1}{|U|} \int_U QW(Du) < QW(F).$$

Can assume U = a polygon + u is piecewise linear on some triangulation (by a standard approx thm). Then (using defn of QW, and our "numerically-oriented discn" we easily get a test fn $\tilde{u}$ st $\tilde{u} = F \cdot x$ on ∂U and

$$\frac{1}{|U|} \int_U W(D\tilde{u}) < QW(F).$$

This is a contradiction – so $Q(QW) = QW$.

For 2nd bullet:

It's easy to see that W not quasiconvex $\Rightarrow$ W not lsc. For example, ~~use~~ periodic changes and suppose Du is periodic on $[0,1]^n$ with $⨍ Du = F$ and $\int_{[0,1]^n} W(Du) < W(F)$. Then

$$u_N(x) = \frac{1}{N} u(Nx)$$

has $⨍ Du_N = F$ and $Du_N(x) = Du(Nx)$ is periodic with period $1/N$. So u_N converges wkly to an affine $u_\infty = F\cdot x + \text{const}$. But $\int_{[0,1]} W(Du_\infty) = W(F) > \lim_{N\to\infty} \int_{[0,1]^n} W(Du_N)$.

So $\int W(Du)$ is not lsc.

Converse is more subtle. For full proof see Dacorogna's book pp 158-166 (this result is also included in Jost + Li-Jost chap 5). Goal is to show W quasiconvex $\Rightarrow$ W lsc, ie
$u_j \rightharpoonup u_\infty$ wkly $W^{1,p}$ $\Rightarrow$ $\int_\Omega W(Du_\infty) \le \liminf_{j\to\infty} \int W(Du_j)$.

Ideas: a) if u_∞ were affine + $u_j|_{\partial\Omega}$ had affine bc then conclusion would be trivial (from defn of quasiconvexity).

b) if u_∞ is affine but we only know $u_j \to u_\infty$ wkly $W^{1,p}$ then we can still say $u_j|_{\partial\Omega} \to$ affine map (since bndry trace is cpt) in suitable norms. Can show, as consequence, that u_j can be modified to have affine bdry cond $\forall j$, while changing the "energy" just a little. Now use (a)

c) any u_∞ can be approximated by a piecewise affine map, with almost the same "energy".

Now use (b).

Suggested exercises:

(1) Show that

$$\inf_{u = F\cdot x \text{ at } \partial\Omega} \int_\Omega W(Du) + |u - F\cdot x|^2 \, dx$$

achieves its minimum if and only if W is quasiconvex.

(2) Show that a quadratic form

$$W(Du) = \sum a_{i\alpha j\beta} D_\alpha u^i D_\beta u^j$$

is quasiconvex if and only if

$$\sum a_{i\alpha j\beta} \xi_i \xi_j \eta_\alpha \eta_\beta \geq 0$$

for all $\xi \in \mathbb{R}^n$ + $\eta \in \mathbb{R}^m$ (here $u: \mathbb{R}^m \to \mathbb{R}^n$).
[Hint: use the characterization involving periodic bc, and Plancherel's formula.]