Calculus of Variations - Lecture 4 - 10/7/09

[We'll start on 10/7 with pp 5-10 of the "Lecture 3" notes. Note there's a minor error at top of pg 5; eqn should be

$$\frac{\partial F}{\partial u_j} - \frac{\partial^2 F}{\partial \dot{u}_j \partial t} - \sum_k \frac{\partial^2 F}{\partial \dot{u}_j \partial u_k} \dot{u}_k = \sum_k \frac{\partial^2 F}{\partial \dot{u}_j \partial \dot{u}_k} \ddot{u}_k$$

but the essential argument is correct.]

Topics in these notes:

a) The example $\displaystyle \min_{\substack{u(0)=0 \\ u(1)=1}} \int_0^1 (u^3 - x)^2 u_x^6 \, dx$

b) discuss how some 1D results do/don't carry over to multiple dimensions.

About (a) [source: Buttazzo, Giaquinta, + Hildebrandt §4.3], key pts are

① min value seems to be 0, achieved by $u(x) = x^{1/3}$

yet

② min value among Lipschitz fns on (0,1) is bounded away from 0 ("the Lavrentiev phenomenon") also

③ if φ is smooth with cpt spt, value of fnl at $u_\varepsilon(x) = x^{1/3} + \varepsilon\varphi$ is typically infinite.

We'll prove ② presently. Assertion ③ is elementary since

$$\left[(x^{1/3} + \varepsilon\varphi)^3 - x \right]^2 \left[x^{1/3} + \varepsilon\varphi \right]_x^6$$

$$= \left(3x^{2/3}\varepsilon\varphi + \text{pos terms} \right)^2 \cdot \left(\tfrac{1}{3}x^{-2/3} + \varepsilon\varphi_x \right)^6$$

$$\geq C\varepsilon^2 x^{4/3}\varphi^2 \cdot x^{-4} \quad \text{near } x=0 \text{ if } \varphi, \varphi_x > 0 \text{ there}$$

which is not integrable near 0.

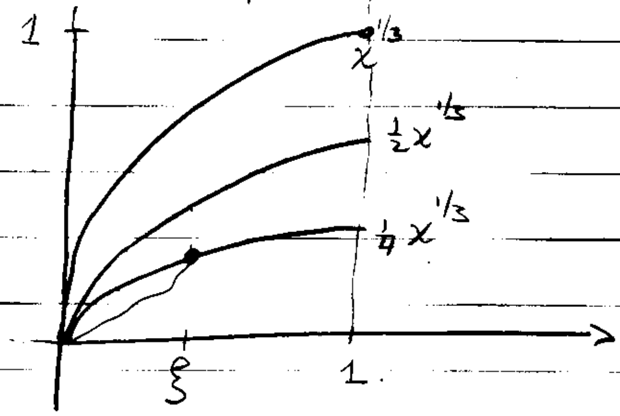
Impact of ③: we have no right to expect EL to hold in such a case (though actually it does hold at $u = x^{1/3}$, by inspection)

Impact of ① - ②: one must choose whether we want the inf over Lip fns (even if it's not achieved) or the min over cont's fns (they're different). [Most numerical schemes will give the former!]

Pt ③ raises the question: when is the EL valid? Answer: when $c(|u_x|^p - 1) \leq F(x, u, u_x) \leq c'(|u_x|^p + 1)$ since then finiteness of integral insures that

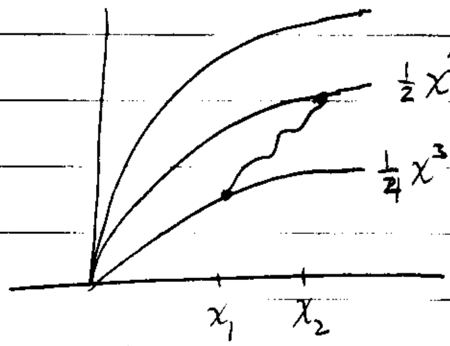
$u \in W^{1,p}$ (and $u + \varepsilon\varphi \in W^{1,p}$ for all ε near 0, and value of variational pbm is diff'ble in ε).

Sketch pf of ②: if u is Lip cont's & $u(0) = 0$ then graph of u is initially below that of $\frac{1}{4}x^{1/3}$.



Let ξ be last pt where graph crosses $\frac{1}{4}x^{1/3}$.

case 1 Suppose $\xi > 1/2$. Then graph includes a piece $x_1 < x < x_2$ on which u stays btwn $\frac{1}{4}x^{1/3}$ + $\frac{1}{2}x^{1/3}$, with $u(x_1) = \frac{1}{4}x_1^{1/3}$, $u(x_2) = \frac{1}{2}x_2^{1/3}$, $\frac{1}{2} \le x_1 < x_2 < 1$.

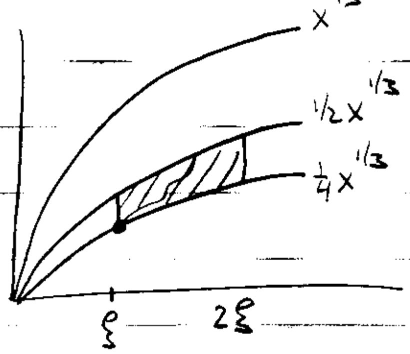


Elementary calcn $\Rightarrow$ $(u^3 - x)^2 \ge \text{const} > 0$ (indep of x_1 + x_2). So value of integral $\geq$ const $\int_{x_1}^{x_2} u_x^6$

and RHS is achieved by lin fn $u_x = \dfrac{u(x_2) - u(x_1)}{x_2 - x_1}$.

arithmetic $\Rightarrow$ pos lower bd.

<u>case 2</u> Suppose $\xi < 1/2$. Then graph leaves region $\xi < x < 2\xi$, $\frac{1}{4}x^{3} < y < \frac{1}{2}x^{3}$ either at RHS or at top.



Elementary calcn $\Rightarrow$ $(u^3-x)^2 \geq c_0\xi^2$ on shaded region.
So value of integral is

$$\geq c_0 \xi^2 \int_{\xi}^{\text{departure pt}} u_x^6$$

If exit is at 2ξ then RHS has lower bd

$$c_0\xi^2 \left(\frac{\Delta u}{2\xi - \xi}\right)^6 \xi \sim \xi^2 \cdot \frac{(\xi^{1/3})^6}{\xi^6}\xi \sim \xi^{-1}$$

If exit is at top a similar calculation (slightly more involved) gives a similar estimate.

Conclusion: in all cases value of integral is bdd below (by a bd indep of u, or even its Lipschitz constant).

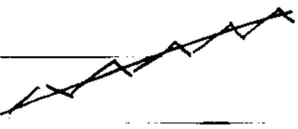
Now topic (b). Defn of "conjugate pt" is strictly a 1D idea.

But what about our arg't that if $u(x)$ is a crit pt with pos 2nd varn then $\partial^2 F / \partial \dot{u}_i \partial \dot{u}_j \geq 0$?

Recall: key to pf was constrn of convenient variation  $\dot{\varphi} = \pm \vec{\xi}$ on a small interval, where ξ has $\partial^2 F / \partial \dot{u}_i \partial \dot{u}_j \, \xi_i \xi_j < 0$.

Also recall: convexity was similarly required for lower semicontinuity. For example if $\xi = \lambda \xi_1 + (1-\lambda) \xi_2$ + $W(\xi) > \lambda W(\xi_1) + (1-\lambda) W(\xi_2)$ Then

$$\min \int_0^1 W(\vec{u}_x) + |\vec{u} - \vec{\xi} x|^2 \, dx.$$

is zero, but it's not achieved (a min seq looks like  slopes $\xi_1 + \xi_2$; avg slope ξ)

This raises the question: given (matrices or vectors) ξ_1 + ξ_2, can we find a fn. u with avg slope $\lambda \xi_1 + (1-\lambda)\xi_2$ but locally $Du = \xi_1$ or ξ_2 a.e.?

<u>Ans</u>: yes for $u: \mathbb{R} \to \mathbb{R}^n$ (obvious)
yes for $u: \mathbb{R}^n \to \mathbb{R}$. (see below)
only if $\xi_2 - \xi_1$ has <u>rank</u> <u>one</u> if $u: \mathbb{R}^n \to \mathbb{R}^m$.

<u>Expln</u>:
If Du jumps from ξ_2 to ξ_1 across Γ + u is cont's there then $D_{tan} u$ is same on both sides $\Rightarrow$ $\xi_2 \cdot \vec{t} = \xi_1 \cdot \vec{t}$ for all ~~tangent~~ vectors t $\Rightarrow$ $\xi_2 - \xi_1 = a \otimes \vec{n}$ where $\vec{n} \perp \Gamma$.

If ξ_1 + ξ_2 are vectors this just means Γ must be $\perp$ to $\xi_2 - \xi_1$, eg in $\mathbb{R}^2$

$Du = \binom{0}{1}$ or $\binom{0}{-1}$ requires layers horizontally

$Du = \binom{1}{0}$ or $\binom{-1}{0}$ requires layers vertically

Consequence: convexity is the natural cond'n (wrt ∇u) for $\int_\Omega W(x, u, \nabla u)\, dx$ if u is <u>scalar</u> valued. But the analogous cond'n in vector-valued setting is <u>rank-one</u> convexity

$$\sum \frac{\partial^2 W}{\partial \nabla_i u^j \partial \nabla_k u^l} \xi_i \eta_j \xi_k \eta_l \geq 0.$$

A different question: can we expect <u>solns</u> to be smooth, <u>in general</u>? Ans is no, <u>even for convex</u> pbms of form

$$\min_{bc} \int_\Omega W(Du) \qquad u : \mathbb{R}^n \to \mathbb{R}^m$$

(1st counterexample was given by <u>Necas</u>, by constructing a convex W st $u_{ij}(x) = \frac{x_i x_j}{|x|}$ was the minimizer for its bc).

However answer is yes when $u : \mathbb{R}^2 \to \mathbb{R}^m$ + W is unif convex (This is a deep thm of Morrey). Also yes when $u : \mathbb{R}^n \to \mathbb{R}^1$ + W is unif convex (This is a deep thm of De Giorgi + Nash).

Is rank-one convexity <u>suff't</u> for lower semicont'y? Ans is <u>no</u> (counterexample was given by Sverak, only abt 1992). We'll return to this later.

Exercise: Let u be a critical pt of $\int_\Omega W(\nabla u)\,dx$ (ie it solves the EL eqn). Assume u is scalar-valued and C^2, and suppose W is nonconvex at $\nabla u(x)$ for some $x \in \Omega$. Show that there exists φ with compact spt in Ω st

$$\frac{\partial^2}{\partial \varepsilon^2}\bigg|_{\varepsilon=0} \int_\Omega W(\nabla(u+\varepsilon\varphi))\, dx < 0 .$$