Calculus of Variations - Lecture 3 - 9/30/09

Today's topic: 1D problems

$$\int_a^b F(t, u(t), \dot{u}(t))\, dt \qquad u : [a,b] \to \mathbb{R}^n$$

emphasizing
(a) geodesics, as a key example
(b) importance of F being convex wrt $\dot{u}$
(c) role of 2^{nd} variation; conjugate pts
(d) regularity of minimizers

Today's material is all in Jost + Li-Jost
Sections 1.1 - 1.3 and 2.1

Key example: <u>geodesics</u>. By defn: a geodesic is a curve that (locally) minimizes arc length. In local coordinates, if the curve is $\vec{x}(t)$,

$$|dx| = |\dot{x}(t)|\, dt = \left(\sum g_{ij}(x(t))\, \dot{x}_i \dot{x}_j\right)^{1/2}$$

where g_{ij} is the Riemannian metric, and the assoc var'l pbm is

$$L = \int_a^b |\dot{x}(t)|\, dt .$$

Two issues:

(1) This has the form $\int F(x(t), \dot{x}(t))\,dt$ but F is not smooth at $\dot{x} = 0$

(2) arc length is indep of parametrization, so it chooses the "curve" but not any specific parametrization

Both issues can be fixed by considering instead the *different* functional

$$E = \frac{1}{2}\int_a^b |\dot{x}|^2\,dt$$

where $|\dot{x}|^2 = \sum g_{ij}(x(t))\,\dot{x}_i(t)\,\dot{x}_j(t)$. To see why, observe that for any parametrized curve $x(t)$

$$L[x] \leq \sqrt{2(b-a)}\ \sqrt{E[x]}$$ with strict inequality unless $|\dot{x}|$ is constant

since

$$\int_a^b |\dot{x}|\,dt \leq \left(\int_a^b |\dot{x}|^2\,dt\right)^{1/2}\left(\int_a^b 1\,dt\right)^{1/2}.$$

Thus

$$\text{min value of } L \leq \sqrt{2(b-a)} \cdot (\text{min value of } E)^{1/2}.$$

But the opposite $\neq$ is easy: given any curve with length l, its constant-speed parzn has $|\dot{x}| = \frac{l}{b-a}$ so

That $\frac{1}{2}\int_a^b |\dot{x}|^2 \, dt \le \frac{1}{2}(b-a)\, \ell^2/(b-a)^2 = \frac{1}{2(b-a)} \ell^2$. So

$$\left(\text{min value of } E\right)^{1/2} \le \frac{1}{\sqrt{2(b-a)}} \left(\text{min value of } L\right)$$

Conclusion: minimizer of E has min length and constant speed.

(Exercise: use the E-L eqn for E to give a different proof that extremals have constant speed, by showing that $\frac{d}{dt}|\dot{x}(t)|^2 = 0$.)

Key properties of geodesics:

(a) they're smooth

(b) they're locally paths of shortest length

(c) globally, they need not be paths of shortest length; for example: on sphere geodesics are arcs of great circles.

Rmk: discn above assumed we had a single "coord chart" valid along the entire geodesic. Locally true, but not necessarily globally so. If necessary can use different coord charts on different parts of curve (see §2.1 of Jost + Li-Jost for more detail on this).

The properties (a) - (c) of geodesics are not *special* to geodesics; Therefore it's natural to discuss them more generally, for var'l problems of form

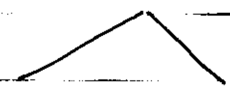
$$\int_a^b F(t, u(t), \dot{u}(t))\, dt \qquad u : [a,b] \to \mathbb{R}^n.$$

Note that EL eqn in this ~~setting~~ is

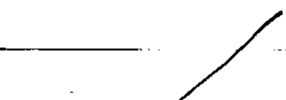
$$\frac{\partial F}{\partial u_j} - \frac{d}{dt}\frac{\partial F}{\partial \dot{u}_j} = 0$$

About (c) = smoothness of solns: we clearly need some hypothesis on F, since when $u : [-1,1] \to \mathbb{R}$,

$$\min_{\substack{u(-1)=0 \\ u(1)=0}} \int_{-1}^{+1} (u_t^2 - 1)^2\, dt \quad \text{is solved by}$$



$$\min_{\substack{u(-1)=0 \\ u(1)=1}} \int_{-1}^{+1} (u_t - 1)^2 u^2\, dt \quad \text{is solved by}$$



Convenient hypothesis is that $F(t,u,p)$ is smooth enough (I won't try to give minimal conds - see Jost + Li-Jost for that) and *strictly convex* w.r.t. p. The pt: then EL eqn can be written

$$\frac{\partial F}{\partial u_j} = \sum_k' \frac{\partial^2 F}{\partial \dot{u}_j \partial \dot{u}_k} \ddot{u}_k$$

which we can solve (inverting the str. pos. def. matrix

$$\frac{\partial^2 F}{\partial \dot{u}_j \partial \dot{u}_k}(t, u(t), \dot{u}(t)) \quad)$$

to see that $\ddot{u}$ is bdd. Higher derivs can be handled similarly.

Preceding arg't is a bit careless, since it assumes $\ddot{u}$ exists. Let's explain why convexity implies it must exist: consider

$$\varphi_j(t, u, p, q) = \frac{\partial F}{\partial p_j} - q_j$$

Recall that p solves $\vec{\varphi}(t, u, p, q) = 0$ (with the other vars held fixed) iff p achieves

$$\max_p \langle q, p \rangle - F(t, u, p, q)$$

Strict convexity of F wr to $p \Rightarrow$ there's a unique such p. Implicit fn thm $\Rightarrow$ we can solve (locally) the eqn $\vec{\varphi} = 0$ for p as a fn of the other vars

$$\frac{\partial F}{\partial p_j} - q_j = 0 \quad \forall j \quad \Rightarrow \quad p_j = \psi_j(t, u, q)$$

But we know $\vec{\varphi} = 0$ when $\vec{q} = \frac{\partial F}{\partial \vec{p}}$ and $p = \dot{u}$. So

$$\dot{u} = \Psi(t, u, \tfrac{\partial F}{\partial \dot{u}}(t, u, \dot{u}))$$

RHS is differentiable in t. Therefore so is LHS.

About (b) = soln being locally minimal: Key tool is 2nd variation. Given a soln of EL eqn, it's natural to consider

$$\frac{d^2}{ds^2}\Big|_{s=0} \int_a^b F(t, u+s\eta, \dot{u}+s\dot{\eta})\, dt$$

which reduces to

$$Q[\eta] = \int_a^b F_{uu}\, \eta \otimes \eta + 2F_{up}\, \eta \otimes \dot{\eta} + F_{pp}\, \dot{\eta} \otimes \dot{\eta} \quad dt$$

where for example

$$F_{uu}\, \eta \otimes \eta = \sum \frac{\partial^2}{\partial u_i \partial u_j} F(t, u, \dot{u}) \cdot \eta_i(t)\, \eta_j(t).$$

Focusing on case when $u(a)$ & $u(b)$ are fixed (so $\eta(a) = \eta(b) = 0$), we see that

2nd varn > 0 if $Q[\eta]$ is strictly positive for all η s.t. $\eta(a) = \eta(b) = 0$.

If $F_{pp} \geq c_0 I$ with $c_0 > 0$ (slightly stronger than strict convexity) then

$$F_{pp}\, \dot{\eta} \otimes \dot{\eta} \geq c_0 |\dot{\eta}|^2$$

and it's easy to show that Q is strictly positive if $|b-a|$ is small enough. (Hint: $\int_a^b |\eta|^2 \leq \lambda(b-a)^2 \int_a^b |\dot{\eta}|^2$ if $\eta(a) = \eta(b) = 0$.)

<u>Convexity condition is natural</u>, since if F is <u>not</u> convex in p then 2nd var'n is negative for a suitable choice of η. In fact, suppose

$$p \to F(t_0, u(t_0), p) \quad \text{is not strictly convex near } p = \dot{u}(t_0)$$

ie (well, I guess this is slightly stronger) suppose $\exists\, \xi$ s.t.

$$\frac{\partial^2 F}{\partial p_i \partial p_j}(t_0, u(t_0), \dot{u}(t_0))\, \xi_i \xi_j < 0.$$
↑ strict!

Then choose $\eta(t)$ s.t. $\dot{\eta}(t) \in \{0, \pm \xi\}$, supported on a nbhd of t_0:

$$\dot\eta = \begin{cases} \xi & t_0-\varepsilon<t<t_0 \\ -\xi & t_0<t<t_0+\varepsilon \end{cases}$$

If ε is small enough then 2^{nd} varn in dirn η is strictly negative (since term $\int F_{pp}\,\dot\eta\otimes\dot\eta\,dt$ scales like ε and is negative; other terms of Q of order ε^2.)

About (c): local optimality can be lost as b increases with a held fixed.

Recall example of a great circle on a sphere.

In general, we can ask whether EL for

$$\min_{\eta(a)=\eta(b)=0} \int_a^b F_{uu}\,\eta\otimes\eta + 2F_{up}\,\eta\otimes\dot\eta + F_{pp}\,\dot\eta\otimes\dot\eta\,dt$$

has a nonzero solution $\eta(t)$. (Such an η solves a "homogeneous" 2^{nd}-order ODE

$$F_{uu}\,\eta + F_{up}\,\dot\eta - \frac{d}{dt}\left(F_{up}\,\eta\right) - \frac{d}{dt}\left(F_{pp}\,\dot\eta\right) = 0$$

and is called a "Jacobi field".) If this is the case we say b is "conjugate to a".

<u>Thm</u>: After 1st conjugate pt, an extremal (i.e. a soln of EL eqn) ceases to be a minimizer.

<u>Pf</u>: Let b_0 be conjugate to a, and $b > b_0$. Try

$$\eta = \begin{cases} \text{nonzero Jacobi field on } (a, b_0) \\ 0 \quad \text{on } (b_0, b) \end{cases}$$

Notice that $F_{uu}\,\eta\otimes\eta + 2F_{up}\,\eta\otimes\dot\eta + F_{pp}\,\dot\eta\otimes\dot\eta = \varphi(t,\eta,\dot\eta)$ is quadratic in η, so

$$\varphi(t, \alpha\eta, \alpha\dot\eta) = \alpha^2\varphi(t,\eta,\dot\eta)$$

$$\Rightarrow \quad \eta\cdot\varphi_\eta + \dot\eta\cdot\varphi_{\dot\eta} = 2\varphi \qquad \text{(by diff'n wrt } \alpha \text{ at } \alpha = 1)$$

$$\Rightarrow \quad \int_a^b \varphi(t,\eta,\dot\eta) = \int_a^{b_0} \varphi(t,\eta,\dot\eta)$$

$$= \frac{1}{2}\int_a^{b_0} \eta\varphi_\eta + \dot\eta\varphi_{\dot\eta}\; dt$$

$$= \frac{1}{2}\int_a^{b_0} \eta\left(\varphi_\eta - \frac{d}{dt}\varphi_{\dot\eta}\right) dt \quad \text{since } \eta(a)=0,\ \eta(b_0)=0$$

$$= 0 \quad \text{since } \eta \text{ solves } \underline{\text{EL eqn}} \text{ for } \int\varphi(t,\eta,\dot\eta)$$

Thus 2nd varn fnl evaluated at $\eta = \underline{0}$. But

η is not C^2 $\Rightarrow$ it cannot be optimal for the 2nd var'n fnl. (Note that $\varphi_{\dot\eta\dot\eta} = F_{pp}$ is assumed to be pos definite!). So 2nd var'n is neg at some $\eta(t)$.

In special case of a 2-sphere, the antipodal pt is the conjugate pt. To see why, observe that

a) for antipodal pts, there's a 1-par family of shortest geodesics (great semicircles).

b) if there's a 1-par family of minimizers $u^\theta(t)$ then $\eta = \frac{\partial}{\partial\theta} u^\theta$ is a Jacobi field.

Pf (a) is obvious. For (b): u^θ solves EL eqn

$$F_u(t, u^\theta, \dot u^\theta) = \frac{d}{dt} F_p(t, u^\theta, \dot u^\theta)$$

Diff in θ $\Rightarrow$

$$F_{uu}\eta + F_{up}\dot\eta = \frac{d}{dt}\left(F_{pu}\eta + F_{pp}\dot\eta\right)$$

which is the eqn describing a Jacobi field.
(More conceptual proof: value of fnl indep of θ $\Rightarrow$ 2nd var'n wrt θ must be zero $\Rightarrow$ $\eta = \frac{d}{d\theta}u^\theta$ achieves value 0 in 2nd var'n fnl $\Rightarrow$ it minimizes this fnl.)

Exercises:

(1) Show directly (using the EL eqn) that any extremal for $\int_a^b |\dot{x}|^2 dt$ has constant speed. Here $|\dot{x}|^2 = \sum g_{ij}(x(t))\, \dot{x}_i(t)\, \dot{x}_j(t)$ (see pp 2-3).

(2) Show that if b is conjugate to a then

$$\min_{\substack{\eta(a)=0 \\ \eta(b)=0}} \int_a^b F_{uu}\, \eta \otimes \eta + 2 F_{up}\, \eta \otimes \dot{\eta} + F_{pp}\, \dot{\eta} \otimes \dot{\eta} = 0$$

(3) When studying waves it is useful to consider paths that minimize travel time, where the wave speed $\gamma(x)$ is a specified function of location x. Show that this amounts to considering geodesics in the metric $g_{ij}(x) = \frac{1}{\gamma(x)} \delta_{ij}$.

(4) In this lecture we focused on Dirichlet bc, ie $\min \int_a^b F(t, u(t), \dot{u}(t))\, dt$ subject to $u(a) = \alpha$ + $u(b) = \beta$ being given. Suppose instead we impose $u(a) = \alpha$ + $u(b) \in M$ where M is a submanifold. What end condition does the EL get at $t = b$? What is the proper notion of a Jacobi field in this case?

(5) Show that the only critical pts of

$$\int_a^b u_x^2 + (u^2-1)^2$$

(no boundary condition!) with nonnegative second variation are the "trivial ones", namely $u \equiv -1$ and $u \equiv +1$. (Hint: let u be a critical point. Show that $\eta = u_x$ achieves value 0 in the 2nd var'n quadratic form. Then show that this η can't be a minimizer of that quadratic form.)