Calculus of Variations - Lecture 13 - 12/9/09.

(Last lecture!) Today:

a) proof of basic homogn Thm, by "Tartar's method" (short + elegant)

b) proof that $F_\varepsilon = \int_\Omega a(x/\varepsilon)|\nabla u|^2 \xrightarrow[\Gamma]{} \int_\Omega a_* |\nabla u|^2$

(provides another example of Γ-convergence)

c) a little perspective on "bounds on effective moduli" and on links btwn this topic + quasiconvexification

Rigorous convergence result via Tartar's method

Recall notation from Lecture 12:

$$\nabla \cdot \left(a(x/\varepsilon) \nabla u_\varepsilon\right) = f \quad \text{in } \Omega$$
$$u_\varepsilon = \varphi \quad \text{at } \partial\Omega$$

$$a(y) \text{ periodic, measurable}$$
$$0 < c_1 \le a(y) \le c_2 < \infty$$

Want to show $u_\varepsilon \to u_*$ and $a(x/\varepsilon)\nabla u_\varepsilon \to a_* \nabla u_*$ where u_* solves "homogenized pbm" as discussed in Lecture 12.

Easy to see that $\int_\Omega |\nabla u_\varepsilon|^2$ is unif bdd (indep of ε).

So the two families

$$\{\nabla u_\varepsilon\}$$

$$\{\sigma_\varepsilon = a(x/\varepsilon)\nabla u_\varepsilon\}$$

both stay unif bdd in $L^2(\Omega; \mathbb{R}^n)$

Passing to subseq if nec + using Rellich's lemma, $\exists$ limits

$$u_0 = \lim_{\varepsilon_j \to 0} u_{\varepsilon_j} \qquad \text{(strongly } L^2\text{, wkly } H^1\text{)}$$

$$\sigma_0 = \lim_{\varepsilon_j \to 0} \sigma_{\varepsilon_j} \qquad \text{(wkly } L^2\text{)}$$

<u>We must show that</u> $\sigma_0 = a_* \nabla u_0$. (This will determine u_0 uniquely - namely as soln of homogenized pde - since it's clear that $\operatorname{div} \sigma_0 = f$. So finally the passage to a subsequence was unnecessary.)

[<u>Note</u>: This arg't gives convergence but no rate. In fact, to get a rate we need some reg'y hypothesis on $a(y)$ + argument proceeds very differently, showing that $\nabla u_\varepsilon = \nabla u_* + \sum \varepsilon \chi^j(\frac{x}{\varepsilon}) \frac{\partial u_*}{\partial x_j}$ + "error term"]

Key idea is this: fixing any vector ξ, we'll show $\langle \sigma_0, \xi \rangle = \langle a_* \nabla u_0, \xi \rangle$ (as fns of x) by making good use of the fn

$$v_\varepsilon(x) = \xi \cdot x + \varepsilon \chi^\xi(x/\varepsilon)$$

(notation as in Lecture 12: $\chi^{\xi} = \sum_j \chi^j \xi^j$). We have (by defn of v_ε + χ^j) that

$$\nabla v_\varepsilon = \xi + \nabla \chi^{\xi}(x/\varepsilon)$$

$$\eta_\varepsilon =_{\text{defn}} a(x/\varepsilon)\nabla v_\varepsilon = a\left(\tfrac{x}{\varepsilon}\right)\xi + a\left(\tfrac{x}{\varepsilon}\right)\nabla\chi^{\xi}\left(\tfrac{x}{\varepsilon}\right)$$

satisfy

$$v_\varepsilon \to \xi\cdot x \quad \text{wkly } H^1, \text{ strongly } L^2$$

$$\eta_\varepsilon \to a_* \xi \quad \text{wkly } L^2, \text{ with } \operatorname{div}\eta_\varepsilon = 0.$$

Assembling this: we have

<u>two sequences of fns</u>	<u>two sequences of vf's</u>
$u_\varepsilon \to u_0$ (unknown)	$a(\frac{x}{\varepsilon})\nabla u_\varepsilon = \sigma_\varepsilon \to \sigma_0$ (unknown)
$v_\varepsilon \to \xi \cdot x$	$a(\frac{x}{\varepsilon})\nabla v_\varepsilon = \eta_\varepsilon \to a_*\xi$
(wk H^1, str L^2)	(wk L^2, divergence-controlled)

Claim that under these hypotheses:

$$\langle \sigma_0, \xi\rangle = \lim_{\varepsilon\to 0}\langle \sigma_\varepsilon, \nabla v_\varepsilon\rangle \quad \text{as distrns} \qquad (*)$$

$$= \lim_{\varepsilon\to 0}\langle a(x/\varepsilon)\nabla u_\varepsilon, \nabla v_\varepsilon\rangle$$

$$= \lim_{\varepsilon\to 0} \langle \nabla u_\varepsilon, a(x/\varepsilon)\nabla v_\varepsilon \rangle$$

$$= \langle \nabla u_0, a_* \xi \rangle \quad \text{as distrns} \qquad (*)$$

$$= \langle a_* \nabla u_0, \xi \rangle$$

The only nontrivial steps are those marked (*). They're both consequences of

<u>Lemma</u>: If σ_ε are vf's st

$$\sigma_\varepsilon \to \sigma_0 \quad \text{wkly } L^2$$
$$\operatorname{div} \sigma_\varepsilon \to \operatorname{div} \sigma_0 \quad \text{strongly } H^{-1}$$

and if v_ε are fns st

$$v_\varepsilon \to v_0 \quad \text{in } H^1 \text{ (wkly)}$$

then

$$\langle \sigma_\varepsilon, \nabla v_\varepsilon \rangle \to \langle \sigma_0, \nabla v_0 \rangle \quad \underline{\text{as distrns}}$$

ie

$$\int_\Omega \varphi \langle \sigma_\varepsilon, \nabla v_\varepsilon \rangle \, dx \to \int_\Omega \varphi \langle \sigma_0, \nabla v_0 \rangle \, dx$$

for all smooth, cptly sptd φ.

Proof: Here integr by parts:

$$\int_\Omega \varphi \langle \sigma_\varepsilon, \nabla V_\varepsilon \rangle = -\int_\Omega \langle \sigma_\varepsilon, \nabla\varphi \rangle V_\varepsilon - \int_\Omega \varphi (\operatorname{div} \sigma_\varepsilon) V_\varepsilon$$

$$\to -\int_\Omega \langle \sigma_0, \nabla\varphi \rangle V_0 - \int_\Omega \varphi (\operatorname{div} \sigma_0) V_0$$

since product (strong conv) · (wk conv) is weakly convergent.

[Rmk: the "div-curl lemma" is a generalization of the preceding lemma.]

Proof that functionals Γ-converge:

Recall what this means: we must show

① for any $u \in H^1(\Omega)$ $\exists$ u_ε st $u_\varepsilon \to u$ (weakly L^2, strongly H^1) st $u_\varepsilon = u$ at $\partial\Omega$ and

$$\lim_{\varepsilon\to 0} \int_\Omega a(x/\varepsilon) |\nabla u_\varepsilon|^2 \leq \int_\Omega \langle a_* \nabla u, \nabla u \rangle$$

② if $u_\varepsilon \to u$ (weakly L^2, strongly H^1) then

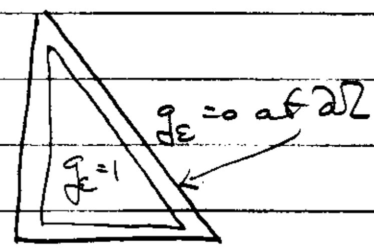
$$\varliminf_{\varepsilon\to 0} \int_\Omega a(x/\varepsilon) |\nabla u_\varepsilon|^2 \geq \int_\Omega \langle a_* \nabla u, \nabla u \rangle$$

Sketch of ①

step 1 for a triangle with affine bc, use constrn we gave in Lecture 12

$$u_\varepsilon = \xi \cdot x + \varepsilon \left(\sum \chi^k(x/\varepsilon)\, \xi^k \right) g_\varepsilon(x)$$

with g_ε = fudge to be sure bc is correct.



note: $|\nabla g_\varepsilon| \sim$ width of fudge layer

step 2 for a piecewise linear fn, use step 1 on each linear "piece"

step 3 any H^1 fn can be approximated by fns that are piecewise linear except on a thin layer near the bdry (not entirely trivial, but a standard result from numerical analysis).

Sketch of ②

Main tool is the following

Lemma: Fix any vector ξ, and recall that (by defn)

$$\langle a_* \xi, \xi \rangle = \min_{\varphi \text{ periodic}} \int_{[0,1]^n} a(y)\,|\xi + \nabla\varphi|^2$$

(I assume $[0,1]^n$ is the period cell for $a(\cdot)$.) I claim that each of the var'l problems

(a) $$\min_{\varphi \text{ periodic}} \int_{[0,N]^n} a(y)\,|\xi + \nabla\varphi|^2$$

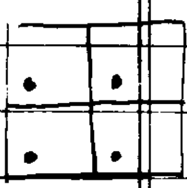
(b) $$\lim_{\varepsilon \to 0} \min_{\varphi = 0 \text{ at bdry}} \int_{[0,1]^n} a(x/\varepsilon)\,|\xi + \nabla\varphi|^2$$

(c) $$\lim_{\varepsilon \to 0} \min_{\varphi = 0 \text{ at bdry}} \int_{\Omega} a(x/\varepsilon)\,|\xi + \nabla\varphi|^2$$

also has value $\langle a_* \xi, \xi \rangle$.

Proof that (a) $= \langle a_* \xi, \xi \rangle$ uses convexity. One way is trivial: any 1-periodic fn is also N-periodic so (a) minimizes over a larger class of φ, whence (a) $\le \langle a_* \xi, \xi \rangle$. For other way, to get idea let's look at 2D case with $N = 2$. Given a 2-periodic φ, consider

$$\tilde\varphi(y_1, y_2) = \tfrac14 \varphi(y_1, y_2) + \tfrac14 \varphi(y_1 + 1, y_2) + \tfrac14 \varphi(y_1, y_2 + 1) + \tfrac14 \varphi(y_1 + 1, y_2 + 1)$$



It is 1-periodic, & $\nabla\tilde{\varphi} = \frac{1}{4}\nabla\varphi(y_1,y_2) + \frac{1}{4}\nabla\varphi(y_1+1,y_2)$
$+ \frac{1}{4}\nabla\varphi(y_1,y_2+1) + \frac{1}{4}\nabla\varphi(y_1+1,y_2+1)$

Since $a(y)$ is 1-periodic, Jensen's ineq gives

$$\frac{1}{4}\int_{[0,2]^2} a(y)\,|\xi+\nabla\varphi|^2 \geq \int_{[0,1]} a(y)\,|\xi+\nabla\tilde{\varphi}|^2 .$$

So min over 1-periodic fns $\leq$ min over 2-periodic fns.
General case (dim n, use N periods) is the same.

Proof that ⓑ $= \langle a_* \xi, \xi\rangle$ is similar to pf that quasiconvexn can be defined using affine or periodic bc. Clearly $\varphi = 0$ at bdry $\Rightarrow$ φ is periodic, so ⓑ $\geq \langle a_*\xi,\xi\rangle$. But if φ achieves optimum in defn of $\langle a_*\xi,\xi\rangle$ then we can get almost same energy with Dir bc by using many periods + introducing a thin bdry layer (as in pf of pt ① of Γ-convergence).

Proof that ⓒ $= \langle a_*\xi,\xi\rangle$ is similar to pf that quasiconvexn defn doesn't depend on the choice of domain: pack $[0,1]^n$ with scaled copies of Ω, or pack Ω with scaled copies of $[0,1]^n$.

QED

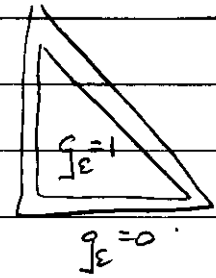
OK, now the rest is easy: suppose $u_\varepsilon \to u$.

step 1 Approximate u by a pl fn (good approxn in H^1 => little effect on energy) off a thin layer near $\partial\Omega$; call the approxn $\tilde{u}$. Note that $u_\varepsilon + (\tilde{u} - u) \to \tilde{u}$ as $\varepsilon \to 0$ (wkly L^2, str H^1).

step 2 Focusing on one "piece", we are left to show that if $u_\varepsilon \to \xi \cdot x$ (strongly L^2, wkly H^1) on a triangle T then

$$\liminf_{\varepsilon} \int_T a(x/\varepsilon)|\nabla u_\varepsilon|^2 \geq \langle a_* \xi, \xi \rangle$$

If u_ε had affine bc this would be consequence of the Lemma (with $\Omega = T$). Otherwise consider



$$\tilde{u}_\varepsilon = u_\varepsilon(x)\, g(x) + (\xi \cdot x)(1 - g(x))$$

Evidently $\nabla \tilde{u}_\varepsilon = \nabla u_\varepsilon \cdot g + \xi(1-g) + (u_\varepsilon - \xi \cdot x)\nabla g$

so

$$\nabla \tilde{u}_\varepsilon - \nabla u_\varepsilon = (\nabla u_\varepsilon - \xi)(g-1) + (u_\varepsilon - \xi \cdot x)\nabla g.$$

We know the desired ineq with u replaced by $\tilde{u}$. So we'll be done once we verify that $\int |\nabla \tilde{u}_\varepsilon - \nabla u_\varepsilon|^2$ can be made as small as one likes (uniformly, for small ε)

by making a suitable choice of g. Term $u_\varepsilon - \xi \cdot x$ is no problem (it $\to 0$ strongly as $\varepsilon \to 0$). Term $(\nabla u_\varepsilon - \xi)(g-1)$ is more subtle: we need

<u>Lemma</u>: take any sequence ε_j s.t. $|\nabla u_{\varepsilon_j}|^2$ converges weakly to a measure μ. Then we can choose g s.t. $\int (g-1)^2 d\mu$ is as small as desired, since an open δ-nbd of ∂T must have measure $\to 0$ as $\delta \to 0$.

Desired assertion follows easily from this.

<u>Rmk</u>: Γ-convergence arg't generalizes even to quasiconvex periodic pbms. But if integrand is not convex then we must in general use <u>many</u> periods in defining eff energy: for u vector-valued, $W(\cdot, \nabla u)$ quasiconvex,

$$\int_\Omega W(x/\varepsilon, \nabla u) \xrightarrow[\Gamma]{} \int_\Omega W_{eff}(\nabla u)$$

where

$$W_{eff}(\xi) = \liminf_{\substack{N \to \infty \\ \varphi \text{ periodic}}} \fint_{[0,N]^n} W(y, \xi + \nabla \varphi).$$

What to ask abt homog'n?

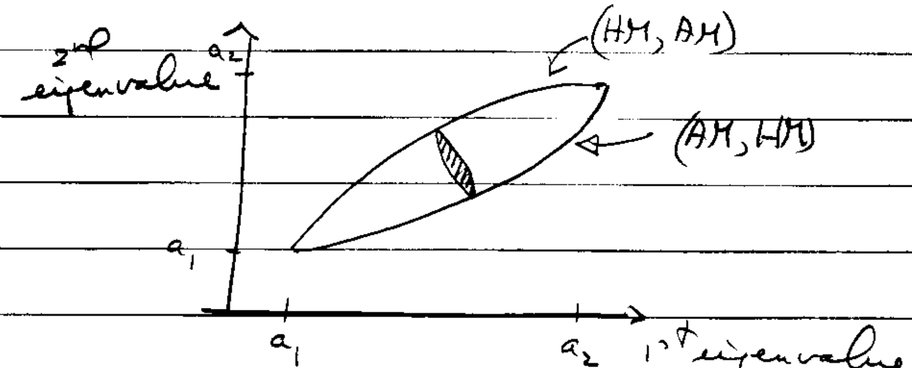
<u>Q1</u>: given microstructure, compute a_*

— Of course, except for special microstrs

The answer must be obtained numerically (we've essentially set this up)

Q2: What matrices a_* are possible, eg by "mixing two materials $a_1 + a_2$ in vol frs $\theta_1 + \theta_2 = 1-\theta_1$" ?

- Surprisingly, answer is completely understood; 2D case.



outer lens gives all a_* possible, as $\theta_1 + \theta_2$ vary; inner lens gives all a_* possible with $\theta_1 + \theta_2$ fixed

inner lens region is bounded by

$$\frac{1}{a_2-\lambda_1} + \frac{1}{a_2-\lambda_2} \le \frac{1}{a_2 - m(\theta)} + \frac{1}{a_2 - h(\theta)}$$

$$\frac{1}{\lambda_1 - a_1} + \frac{1}{\lambda_2 - a_1} \le \frac{1}{m(\theta) - a_1} + \frac{1}{h(\theta) - a_1}$$

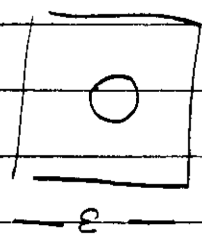
where $m(\theta) = AM = \theta_1 a_1 + \theta_2 a_2$

$h(\theta) = HM = (\theta_1 a_1^{-1} + \theta_2 a_2^{-1})^{-1}$

(See eg book by Gregoire Allaire on "optimal design"; result stated above was proved by Murat + Tartar, + indep by Lurie + Cherkaev, in 80's)

Q3: What if "materials being mixed" can depend on ε = local length scale?

- Then limiting behavior can be very different, eg leading order term in expansion can be oscillatory. Example studied by Bouchitté + Felbacq (as basic example of a "metamaterial"):



period cell

$$a_\varepsilon(x) = \begin{cases} \varepsilon^2 & \text{in inclusion} \\ 1 & \text{outside} \end{cases}$$

$$\nabla \cdot \left(a_\varepsilon(x) \nabla u_\varepsilon \right) + \lambda u_\varepsilon = 0$$

Then $u_\varepsilon \approx u_*(x) \cdot M(x/\varepsilon)$

where $M(y)$ solves $\Delta_y M + \lambda M = 0$ in inclusion,

$M = 1$ at incl bdry + in matrix