Calculus of Variations - Lecture 12 - 12/2/09

New topic today: homogenization of uniformly elliptic pde with periodic microstructure.

Motivation:

(1) another example of Γ-convergence (rather different from Modica-Mortola)

(2) strong links to physics + materials science, as simplest context where we can discuss "effective moduli" of materials with microstructure

Some places to read more:

a) Bensoussan, Lions, Papanicolaou, Asymptotic Analysis for Periodic Structures (a classic)

b) Cioranescu + Donato, An Introduction to Homogenization (more modern)

(Braides' book on Γ-convergence has a chapter on "Some Homogenization Problems" but the focus is on fns of one spatial variable; Jost + Li-Jost have a short section on "Homogenization" but the focus is on many small holes with a Dirichlet bc. Interesting + relevant but different from what I do here.)

<u>What is the problem?</u> Want to think abt "effective" or "macroscopic" behavior of a mixture of 2 (or more) materials. Microstructure will be <u>periodic</u> (for simplicity) though similar theory can be developed for random microstrs, or (through a compactness argument) for microstructures with no spatial organization at all

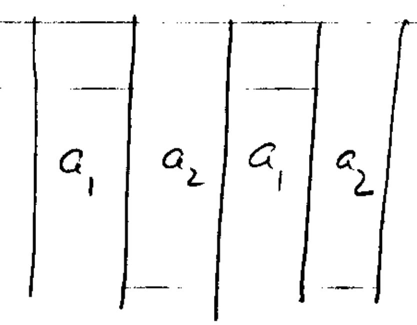
Focus on scalar pde

$$\nabla\cdot\left(a\left(\tfrac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = f \quad \text{in} \quad \Omega$$

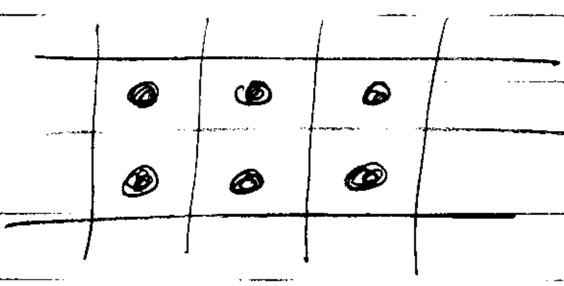
$$u_\varepsilon = \varphi \quad \text{at} \quad \partial\Omega$$

where $a(x/\varepsilon)$ is real-valued + spatially periodic. To fix ideas, let's focus on case where $a(\cdot)$ takes just two values a_1 + a_2 (both positive!).

<u>Examples</u>:



layered microstructure



periodic array of inclusions

Small parameter is $\varepsilon = \dfrac{\text{length scale of microstr}}{\text{length scale of domain + data}}$.

Link to Γ-convergence: u^ε solves

$$F_\varepsilon(u) = \int_\Omega \tfrac{1}{2}\, a(\tfrac{x}{\varepsilon})\, |\nabla u|^2 + u\cdot f \; dx$$

subject to $u = \varphi$ at $\partial\Omega$.

<u>Basic Thm</u>: $\exists$ (constant) matrix a_*, the <u>effective tensor</u> of the composite, st u_ε converges as $\varepsilon \to 0$ to soln u_* of

$$\nabla\cdot(a_* \nabla u_*) = f \quad \text{in } \Omega$$
$$u_* = \varphi \quad \text{at } \partial\Omega$$

Moreover there's a formula for a_*, that depends only on the microstructure (not on f, φ, or Ω):

$$\langle a_* \xi, \xi\rangle = \min_{\varphi \text{ periodic}} ⨍ \langle a(y)(\xi + \nabla\varphi), \xi + \nabla\varphi\rangle$$

where $⨍$ denotes the average of a spatially periodic function. Also: $a_* \xi = ⨍ a(y)(\xi + \nabla\varphi^\xi)$ where φ^ξ is optimal for given ξ.

Note that when $a(y)$ is discont's, eqn

$$\nabla\cdot\left(a\left(\tfrac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = f$$

means — pde holds in classical sense away from discontinuity

— $\left(a\left(\tfrac{x}{\varepsilon}\right)\nabla u_\varepsilon\right)\cdot\nu$ is cont's + u_ε is cont's } across surface where $a(\cdot)$ jumps.

Examples, to build intuition:

① in 1D pde can be solved more or less explicitly

$$\left(a(x/\varepsilon)\, u_{\varepsilon x}\right)_x = f \implies a(x/\varepsilon)\, u_{\varepsilon x} = f + c$$

(c = constant)

$$\implies u_{\varepsilon x} = a^{-1}\left(\tfrac{x}{\varepsilon}\right)(f+c)$$

Note that $u_{\varepsilon x}$ has oscillations of order 1 in magnitude (u_ε therefore has osc of order ε in magnitude)

Lemma that we'll use repeatedly: if $F(x,y)$ is bounded, cont's, + periodic in y then

$$f_\varepsilon(x) = F(x, x/\varepsilon)$$

converges weakly in every L^p $(1<p<\infty)$ to

$$f_0(x) = \int F(x,y)\,dy$$

Pf: easy to see $f_\varepsilon \to f_0$ as distributions. Now use boundedness in L^p + standard compactness results.

[Rmk: hypotheses on $F(x,y)$ can be weakened considerably]

Returning to the example: evidently

$$u_\varepsilon \to u_* \quad \text{(weakly } H^1\text{, strongly } L^2\text{) where}$$

$$u_{*x} = \left[\int a^{-1}(y)\right] (f(x)+c)$$

$$\Rightarrow \quad \partial_x (a_* \partial_x u_*) = 0, \quad a_* = \left(\int a^{-1}\,dy\right)^{-1}$$

NB: if $a(y) = \begin{cases} a_1 & \text{on fraction } \theta_1 \\ a_2 & \text{on fraction } \theta_2 = 1-\theta_1 \end{cases}$ then

$$a_* = \left(\frac{\theta_1}{a_1} + \frac{\theta_2}{a_2}\right)^{-1} \quad \text{"harmonic mean"}$$

② 2D layered case, with layers $\parallel x_2$-axis (ie a is a fn of $\frac{x_1}{\varepsilon}$ in (x_1,x_2) plane)

Then $a_* = \begin{pmatrix} (\fint a^{-1})^{-1} & 0 \\ 0 & \fint a \end{pmatrix} = \begin{pmatrix} HM & 0 \\ 0 & AM \end{pmatrix}$

Note the anisotropy! Let's deduce this from the formula for a_* given by the basic thm:

$\xi = (1,0) \Rightarrow \langle a_* \xi, \xi \rangle = \min \fint a(y_1) \, |\xi + \nabla\varphi|^2$
is solved by taking $a(y_1)(1+\varphi'(y_1)) = \text{constant}$.
(φ periodic $\Rightarrow$ constant must be $(\fint a^{-1})^{-1}$)
$\Rightarrow a_*(1,0) = \fint a(y)(\xi + \nabla\varphi) = ((\fint a^{-1})^{-1}, 0)$

$\xi = (0,1) \Rightarrow \langle a_* \xi, \xi \rangle = \min \fint a(y_1) \, |\xi + \nabla\varphi|^2$
is solved by taking $\varphi \equiv 0$
$\Rightarrow a_*(0,1) = \fint a(y)(0,1) = (0, \fint a)$

[in both cases we recognized the soln by checking that EL eqn was satisfied; by convexity that's suff't]

There are 2 ways to explain what's going on:
a) asymptotic expansion ("the two-scale method")
b) averaging of the var'l principle.

Both represent useful, generalizable methods, so let's do them both.

<u>Two-scale method</u>: look for expansion

$$u_\varepsilon(x) = u_0(x, \tfrac{x}{\varepsilon}) + \varepsilon u_1(x, \tfrac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \tfrac{x}{\varepsilon}) + \cdots$$

where each $u_i(x,y)$ is periodic in its 2nd var "y".

Substitute into eqn: $\frac{\partial}{\partial x_i} \to \frac{\partial}{\partial x_i} + \varepsilon^{-1}\frac{\partial}{\partial y_i}$

$$\sum_i \left[\frac{\partial}{\partial x_i} + \varepsilon^{-1}\frac{\partial}{\partial y_i}\right] a(y) \left(\frac{\partial}{\partial x_i} + \varepsilon^{-1}\frac{\partial}{\partial y_i}\right)(u_0 + \varepsilon u_1 + \cdots) = f$$

Now extract eqns at various orders

<u>order ε^{-2}</u> $\quad \sum \frac{\partial}{\partial y_i}\left(a(y)\frac{\partial}{\partial y_i} u_0(x;y)\right) = 0$

view this as pde for $y \to u_0(x;y)$, with x as parameter.

<u>Fact</u>: if w is periodic in y & $\nabla\cdot(a(y)\nabla w) = 0$ (& $w \in H^1$) then $w \equiv$ constant (ie indep of y).
<u>Pf</u>: multiply eqn by w & integrate by pts, to see $\int a(y)|\nabla_y w|^2 = 0$.

So $u_0 =$ fn of x alone: $\boxed{u_0 = u_0(x)}$

<u>Note</u>: This u_0 is in fact our u_*. But we must proceed to see what pde it solves.

at order ε^{-1}: $\sum' \frac{\partial}{\partial y_i}\left[a(y)\frac{\partial u_0}{\partial x_i}\right] + \frac{\partial}{\partial y_i}\left[a(y)\frac{\partial u_1}{\partial y_i}\right] = 0$

ie $\quad \mathrm{div}_y\left(a(y)\nabla_y u_1\right) = -\mathrm{div}_y\left(a(y)\nabla_x u_0\right)$

This is a pde for $u_1(x,y)$ [with x as a parameter]. Consistency condn is satisfied (RHS has mean 0 wr to y).

Examine pde further: we see dependence on $\nabla_x u_0$ is linear. So if χ^i solves

$$\mathrm{div}_y\left(a(y)\nabla_y \chi^i\right) = -\mathrm{div}_y\left(a(y)\, e_i\right)$$

with $e_i = (0,\cdots,\underset{i^{th}\text{place}}{1},\cdots,0)$, then we can take

$$u_1(x,y) = \sum_i' \chi^i(y)\frac{\partial u_0}{\partial x_i}$$

(up to an additive fn of x alone).

at order 0: this is where we will get eff eqn of course.

$$\sum_i' \frac{\partial}{\partial x_i}\left(a(y)\frac{\partial u_0}{\partial x_i}\right) + \frac{\partial}{\partial y_i}\left(a(y)\frac{\partial u_1}{\partial x_i}\right) + \frac{\partial}{\partial x_i}\left(a(y)\frac{\partial u_1}{\partial y_i}\right)$$
$$+ \frac{\partial}{\partial y_i}\left(a(y)\frac{\partial u_2}{\partial y_i}\right) = f(x).$$

View as pde for u_2 wrt y, with x as parameter, + periodic bc. Consistency cond: data must have avg value 0 wr to y. So

$$\int \frac{\partial}{\partial x_i}\left(a(y)\frac{\partial u_0}{\partial x_i}\right) + \frac{\partial}{\partial x_i}\left(a(y)\frac{\partial u_1}{\partial y_i}\right) dy = 0.$$

Recall that $u_1 = \sum_j \chi^j(y)\frac{\partial u_0}{\partial x_j}(x)$, so

$$a(y)\frac{\partial u_1}{\partial y_i} = \sum_j a(y)\frac{\partial \chi^j}{\partial y_i}\frac{\partial u_0}{\partial x_j}$$

Thus u_0 solves

$$\nabla\cdot\left(a_* \nabla_x u_0\right) = f$$

with

$$(a_*)_{ij} = \int \left[a(y)\delta_{ij} + a(y)\frac{\partial \chi^j}{\partial y_i}\right] dy$$

Rmk: From this viewpt formula for a_* looks rather mysterious. It's not even obvious that $(a_*)_{ij} = (a_*)_{ji}$ – though this is true. Our 2nd, var'l approach will make things clearer.

Variational viewpt (still formal, but organized around var'l prin rather than pde; better because it's more intuitive + leads more easily to var'l prin for a_*.)

Start with ansatz:

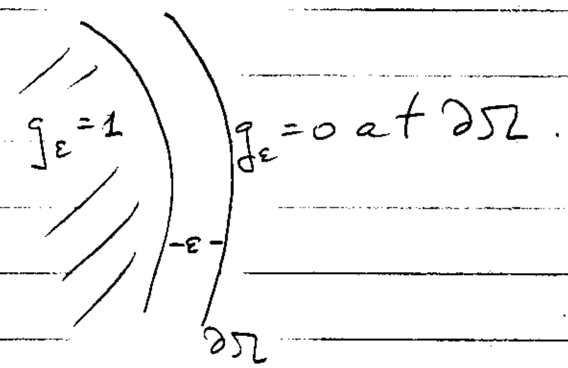
$$u_\varepsilon = u_0(x) + \varepsilon\left(\sum_k \chi^k\left(\tfrac{x}{\varepsilon}\right)\frac{\partial u_0}{\partial x_k}\right) g_\varepsilon(x)$$

where

u_0 = unknown fn of x (satisfying bc of pde at $\partial\Omega$)

$\chi^k(y)$ = unknown periodic fn of y

$g_\varepsilon(x)$ = fudge to be sure u_ε has proper bc.



(Motivation: 2-scale exp. at order ε^{-1} told us, with a special choice of χ^k; here we keep the ansatz but leave χ^k to be determined later.)

Substitute into var'l prin $\Rightarrow$

$$\nabla_x u_\varepsilon = \nabla_x u_0 + \sum{}' \nabla_y \chi^k\left(\tfrac{x}{\varepsilon}\right) \frac{\partial u_0}{\partial x_k} + \text{error terms}$$

(where "error terms" are <u>either</u> of magnitude ε, or else of magnitude 1 supported on an ε-strip near bdry). Ignoring error terms,

$$\int a\left(\tfrac{x}{\varepsilon}\right)|\nabla u|^2 = \int a(x/\varepsilon)\left|\nabla_x u_0 + \sum_k \nabla_y \chi^k\left(\tfrac{x}{\varepsilon}\right)\frac{\partial u_0}{\partial x_k}\right|^2$$

RHS has form $F(x,y)\big|_{y=x/\varepsilon}$, which converges weakly to $\int F(x,y)\,dy$. So as $\varepsilon \to 0$, "energy" becomes

$$\int \langle a_* \nabla u_0, \nabla u_0\rangle$$

where

$$(a_*)_{kl} = \int \sum_i{}' a(y)\left(\delta_{ik} + \frac{\partial \chi^k}{\partial y_i}\right)\left(\delta_{il} + \frac{\partial \chi^l}{\partial y_i}\right) dy.$$

Here I used that

$$\left|\nabla_x u_0 + \sum_k{}' \nabla_y \chi^k\left(\tfrac{x}{\varepsilon}\right)\frac{\partial u_0}{\partial x_k}\right|^2 = \sum_{i,k,l}{}' \left(\delta_{ik} + \frac{\partial \chi^k}{\partial y_i}\right)\frac{\partial u_0}{\partial x_k}\left(\delta_{il} + \frac{\partial \chi^l}{\partial y_i}\right)\frac{\partial u_0}{\partial x_l}$$

<u>Explanation of choice of χ^j</u>: for given $\xi = \nabla u_0$, we naturally want to minimize

$$\langle a_* \xi, \xi \rangle = \int \sum_{i,k,l} a(y) \left(\delta_{ik} + \frac{\partial \chi^k}{\partial y_i}\right)\left(\delta_{il} + \frac{\partial \chi^l}{\partial y_i}\right) \xi_k \xi_l$$
$$= \int a(y) \, |\xi + \nabla \chi^\xi|^2$$

with $\chi^\xi = \sum \xi^k \chi^k$. This is our var'l prin for a_*.
Its Euler eqn is

$$\operatorname{div}_y \left(a(y) \left(\xi + \nabla \chi^\xi \right) \right) = 0$$

ie

$$\sum_{i,l} \frac{\partial}{\partial y_i} a(y) \left(\delta_{il} + \frac{\partial}{\partial y_i} \chi^l \right) \xi^l = 0$$

Coefft of ξ^l is precisely the pde we got before, by asymptotic expansion, for χ^l.

Suggested exercises

(1) The two scale method led to

$$(a_*)_{ij} = \int a(y) \left(\delta_{ij} + \frac{\partial \chi^i}{\partial y_j} \right) dy$$

where χ^i solves

$$\sum_k \frac{\partial}{\partial y_k} \left(a(y) \left[\delta_{ik} + \frac{\partial \chi^i}{\partial y_k} \right] \right) = 0$$

Show directly that $(a_*)_{ij} = (a_*)_{ji}$

(2) The following was assigned already when we did convex duality, but I repeat it as a reminder: from the var'l principle for a_*, one can show also that

$$\langle (a_*)^{-1} \eta, \eta \rangle = \min_{\substack{\operatorname{div} \tau = 0 \\ \fint \tau = \eta}} \fint \langle a^{-1}(y) \tau, \tau \rangle \, dy$$

(3) Show that

$$\left(\fint a^{-1}\right)^{-1} |\xi|^2 \le \langle a_* \xi, \xi \rangle \le \left(\fint a\right) |\xi|^2.$$

Thus: the eigenvalues of a_* lie between the arithmetic + harmonic means.

(4) Consider a 2D conducting checkerboard, in which the "red" squares have $a(y) = \alpha$ and the "black" squares have $a(y) = \beta$. Show that $a_* = \sqrt{\alpha\beta}$. (Hint: the "dual" variational principle in problem (2) looks a lot like the original one, if we are in 2D, because we can then take $\tau = \eta + (\nabla \psi)^\perp$ with ψ periodic.)