Calculus of Variations - Lecture 11 - 11/25/09

These notes complete the discn of the "Modica-Mortola problem" and suggest a few exercises.

As explained in Lecture 10, the statement that

$$F_\varepsilon(u) = \int_\Omega \varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}(u^2-1)^2$$

Γ-converges to

$$F_0(u) = \begin{cases} 8/3 \cdot \mathrm{Per}_\Omega\{x : u(x) = 1\} & \text{if } u = \pm 1 \text{ a.e.} \\ \infty & \text{else} \end{cases}$$

in the $L^1(\Omega)$ topology means

① if $v_\varepsilon \to v_0$ in L^1 as $\varepsilon \to 0$, then

$$\liminf F_\varepsilon(v_\varepsilon) \geq F_0(v_0)$$

② if $v_0 \in L^1$ then $\exists\, v_\varepsilon$ st $v_\varepsilon \to v_0$ in L^1 &

$$F_\varepsilon(v_\varepsilon) \to F_0(v_0).$$

It is also impt to know that the fnls are "uniformly coercive" in the sense that bddness of F_ε implies compactness in L^1

③ if $\{v_\varepsilon\}$ have $F_\varepsilon(v_\varepsilon)$ unif bdd, then $\{v_\varepsilon\}$ is compact in L^1 (ie each sequence has a convergent subsequence)

We've already explained the essential ingredients of the proofs, but lets now give more complete arguments for ① + ③ and a bit more guidance on ② as well.

The rigorous arguments use some basic facts about the function space

$$BV(\Omega) = \left\{ u \in L^1(\Omega) \text{ st } \int_\Omega |\nabla u| < \infty \right\}$$

where Ω is bounded and we use the (slightly abusive) notation

$$\int_\Omega |\nabla u| = \sup_{\substack{g \in C_0^\infty(\Omega, \mathbb{R}^n) \\ |g| \le 1}} \int_\Omega u \operatorname{div} g \, dx$$

Here $|\nabla u|$ is in general not an L^1 function, but rather a signed measure; for any cont's $h(x)$,

$$\int_\Omega h |\nabla u| = \sup_{g \in C_0^\infty, |g| \le h} \int_\Omega u \operatorname{div} g .$$

In particular, the characteristic fn of a set with smooth bdry is in BV, and

$$\mathrm{Per}_\Omega A = \int_\Omega |\nabla \chi_A|$$

Some (relatively easy) facts:

A) lower semicontinuity: if $u_\varepsilon \to u$ in $L^1(\Omega)$ then $\liminf_\varepsilon \int_\Omega |\nabla u_\varepsilon| \geq \int_\Omega |\nabla u|$

(This is obvious, since $\int |\nabla u|$ is a sup of cont's linear functionals.)

B) bounded sets in the BV norm ($\|u\|_{BV} = \|u\|_{L^1} + \int |\nabla u|$) are compact in L^1

(This is the BV analogue of the familiar statement that the embedding $W^{1,p} \hookrightarrow L^q$ is compact for Ω = bdd subset of $\mathbb{R}^n$ when $q < \frac{np}{n-p}$. Our case is $q=1$ and [roughly] $p=1$.)

C) if a set A has bounded perimeter (ie its char fn $\chi_A \in BV$) then there are sets $A_k \subset \Omega$ with C^2 bndry st $\chi_{A_k} \to \chi_A$ in L^1 + $\mathrm{Per}_\Omega(A_k) \to \mathrm{Per}_\Omega(A)$ + $\mathcal{H}^{n-1}(\partial A_k \cap \partial\Omega) = 0$.

(This is proved via mollification, using the co-area formula.)

The book by Jost + Li-Jost has complete discuss of (A) + (B), and all the ingredients needed to prove (C).

Now proof of ①:

Observe first that it suffices to consider limits u st $u(x) = \pm 1$ ae, since if $u_\varepsilon \to u$ in L^1 then

$$\liminf \int \varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}(u_\varepsilon^2 - 1)^2 \geq \liminf_\varepsilon \frac{1}{\varepsilon}\int (u_\varepsilon^2-1)^2$$

$$\geq \int \liminf \frac{1}{\varepsilon}(u_\varepsilon^2 - 1)^2$$

$= \infty$ if u takes values other than ± 1 on a set of positive measure

Next, observe that it suffices to consider u_ε st

$$-1 \leq u_\varepsilon \leq +1$$

since otherwise we can replace u_ε by its truncation

$$u_\varepsilon^* = \begin{cases} +1 & u_\varepsilon(x) > 1 \\ u_\varepsilon(x) & -1 \le u_\varepsilon(x) \le +1 \\ -1 & u_\varepsilon(x) < -1 \end{cases}$$

without changing the L^1 limit, and this truncation decreases the value of F_ε.

Finally, suppose $-1 \le u_\varepsilon(x) \le +1$ ae & $u_\varepsilon \to u_0$ in L^1 with $u_0 = \pm 1$ ae. Then

$$F_\varepsilon(u_\varepsilon) \ge 2 \int_\Omega |\nabla \varphi(u_\varepsilon)|$$

where $\varphi(t) = \int_{-1}^{t} |1-t^2|\, dt$ (we explained this in Lecture 10). By dominated convergence, $\varphi(u_\varepsilon) \to \varphi(u_0)$ in L^1. By lsc of the BV norm,

$$\liminf F_\varepsilon(u_\varepsilon) \ge 2 \int_\Omega |\nabla \varphi(u_0)|$$

But

$$2\varphi(u_0) = \begin{cases} 0 & \text{where } u_0 = -1 \\ 8/3 & \text{where } u_0 = +1 \end{cases}$$

So $2 \int_\Omega |\nabla \varphi(u_0)| = \frac{8}{3} \operatorname{Per}_\Omega \{u_0 = 1\}$, QED

Sketch proof of ②: as we explained in Lecture 10, for equality to hold in

$$\int \varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} |u_\varepsilon^2 - 1|^2 \geq 2 \int |\nabla \varphi(u_\varepsilon)|$$

we would need $\varepsilon |\nabla u_\varepsilon| \simeq \frac{1}{\varepsilon} |u_\varepsilon^2 - 1|$. In 1D this is achieved by using $u_\varepsilon \approx \pm \tanh\left(\frac{x - x_j}{\varepsilon}\right)$ near the j^{th} transition (modified far from the transition so $u_\varepsilon = \pm 1$ there). In $\mathbb{R}^n$, $n \geq 2$, you get the same effect (as $\varepsilon \to 0$, with the geometry smooth enough + held fixed) by using

$$u_\varepsilon \approx \tanh\left(\frac{\operatorname{dist}(x, A_k)}{\varepsilon}\right).$$

Since by point C above it suffices to consider smooth sets A_k, this works.

(Making this argument fully precise is somewhat tedious.)

Proof of ③, i.e. uniform coercivity in L^1: recall that we have a unif bd on $\int |\nabla \varphi(u_\varepsilon)|$ where

$$\varphi \text{ is monotone, } \varphi'(t) = |t^2 - 1|$$

and, using the form of φ',

$$|\varphi(t)| \le C(1+|t|^3).$$

It follows that

$$\int_\Omega |\varphi(u_\varepsilon)| \le C\int_\Omega 1+|u_\varepsilon|^3 < \text{const indep of } \varepsilon$$

since $\frac{1}{\varepsilon}\int_\Omega (u_\varepsilon^2-1)^2$ stays bdd, and $(u^2-1)^2 \sim u^4$ when $|u|>>1$,

So the BV norms of $\{\varphi(u_\varepsilon)\}$ stay bdd, whence $\{\varphi(u_\varepsilon)\}$ are compact in L^1. So any seq has a subsequence st

$$v_{\varepsilon_j} = \varphi(u_{\varepsilon_j}) \to v_0 \quad \text{in } L^1$$

By unif cont'y of φ^{-1} we conclude that

$$u_{\varepsilon_j} \text{ converges in measure to } \varphi^{-1}(v_0).$$

Since u_{ε_j} are unif bdd in L^4, it follows that they converge to $u_0 = \varphi^{-1}(v_0)$ in L^1.

Suggested exercises:

(1) Suppose u is vector-valued, $u: \Omega \to \mathbb{R}^n$, and let $W(u)$ be a "two-well potential" such that

$$W \geq 0, \text{ with } W = 0 \text{ exactly at two values } \vec{a} + \vec{b} \in \mathbb{R}^n.$$

Show that

$$\int_\Omega \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \geq \int_\Omega |\nabla \varphi(u)|$$

where $\varphi(z)$ is "the distance from $\vec{a}$ to $\vec{z}$ in the metric with weight $W^{1/2}$," i.e.

$$\varphi(z) = \min_{\substack{y(0)=a \\ y(1)=z}} \int_0^1 W^{1/2}(y(t))\, |\dot{y}(t)|\, dt$$

[Remark: this is the first step in the proof that $\int_\Omega \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \xrightarrow{\Gamma} c_0 \cdot \text{Perimeter}$, where $c_0 = \varphi(b)$, provided the Hessian of W is strictly positive at both $a + b$.]

(2) In the 1D scalar setting, we can replace $(u^2-1)^2$ by a "double-well energy" $W(u)$ such that

$W \geq 0$, with $W = 0$ exactly at $u = a$ + $u = b$

for some $a < b \in \mathbb{R}$. But it's important for property ② that $W''(a) > 0$ and $W''(b) > 0$. Explain why this is the case, by considering the solution of the ODE $u_x = W^{1/2}(u)$. (Hint: $\int_a^b \frac{du}{W^{1/2}(u)}$ can be finite if $W''(a) = W''(b) = 0$.)