# Calculus of Variations – Lecture 10 – 11/18/09

Looking ahead: plan is

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|---|---|
| 11/18, 11/25 | $\Gamma$-convergence |
| 12/2, 12/9, 12/16 | homogenization |

Wed 12/16 is "reading day" but (depending how far we get on 12/9) I may give a lecture that day. If so, it will end in plenty of time for the Holiday Lecture.

New topic today: the "Modica-Mortola problem" (interesting by itself, and as a 1st exposure to $\Gamma$-convergence)

Sources:

- Chapter 7 of Jost + Li-Jost is pretty good (it includes lots of technical stuff I'll skip over) but note: it focuses on a slightly different Modica-Mortola example than my focus here.
- Relatively easy reading, + basics of a lot of what I'll do here: RV Kohn + P. Sternberg, "Local minimizers + singular perturbations," Proc

Roy Soc Edinburgh 111A (1989) 69-84

- For $\Gamma$-conv. even in presence of a vol constraint, see P. Sternberg's article Arch Rat Mech Anal 101 (1988) 209-260 or L Modica's article Arch Rat Mech Anal 98 (1987) 123-142.

- For $\Gamma$-conv. w/o constraint on vol, "Modica-Mortola" problem was discussed by L Modica + S Mortola in Boll. Unione Mat. Italiana - 2 articles in Italian in 1977.

Our goal is to "understand the asymptotic behavior of"

$$F_\varepsilon(u) = \int_\Omega \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon}(u^2-1)^2 \, dx$$

in the limit as $\varepsilon \to 0$ (here $\Omega \subset \mathbb{R}^n$ bdd, + $u: \Omega \to \mathbb{R}$). Answer will be: $F_\varepsilon$ "$\Gamma$-converges" to

$$F_0(u) = \begin{cases} \frac{8}{3} \cdot \mathrm{Per}_\Omega \{x : u(x) = 1\} & \text{if } u = \pm 1 \text{ a.e.} \\ \infty & \text{otherwise} \end{cases}$$

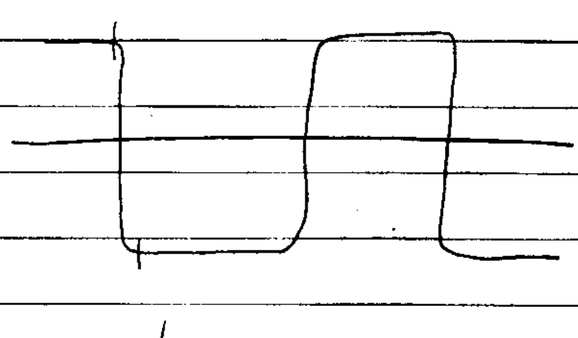
and as consequences

a) if $G(u)$ is compact under $L^1$ convergence, minimizers of $F_\varepsilon(u) + G(u)$ converge to minimizers of $F_0(u) + G(u)$

b) an isolated $L^1$-local min of $F_0(u)$ is the limit as $\varepsilon \to 0$ of $L^1$-local minimizers of $F_\varepsilon$.

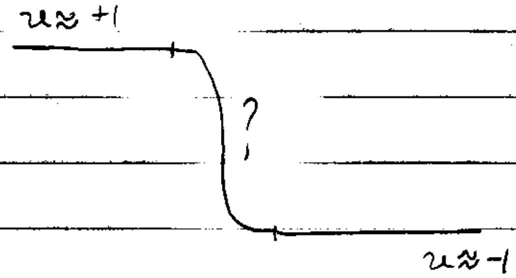
(We'll explain these in due course; but note: the main point of $\Gamma$-convergence is to provide information about minimizers.)

Some intuition first: suppose $\Omega = [0,1] \subset \mathbb{R}$ and $\varepsilon$ is small. If $F_\varepsilon$ is not of order $\frac{1}{\varepsilon}$ then $u^2 \approx 1$ so $u$ should be close to $\pm 1$, except perhaps for some "transitions"



Modica-Mortola asserts: as $\varepsilon \to 0$, energy required for a transition is exactly $8/3$.

It's easy to see intuitively why this should be true: suppose $u \approx +1$ for $x \le a$ + $u \approx -1$ for $x \ge b$.




Then

$$\int_a^b \varepsilon u_x^2 + \frac{1}{\varepsilon}(u^2-1)^2 \ge 2\int_a^b |u^2-1||u_x|$$

$$= 2\int_a^b |\varphi(u)_x| = 2\,|\varphi(1)-\varphi(-1)|.$$

where $\varphi'(t) = |t^2-1|$, eg $\varphi(t) = \int_{-1}^t |1-s^2|\,ds$.
for which

$$-1<t<1 \Rightarrow \varphi(t) = t - \tfrac{1}{3}t^3 \Big|_{-1}^{t} \Rightarrow \varphi(1)-\varphi(-1) = \tfrac{4}{3}.$$

Preceding calcn also shows form of the optimal transition: it has

$$\sqrt{\varepsilon}|u_x| = \frac{1}{\sqrt{\varepsilon}}|u^2-1| \quad \text{ie} \quad \varepsilon u_x = \pm(1-u^2)$$

This is easily integrated, to show that $u \approx \pm\tanh\left(\frac{x-x_0}{\varepsilon}\right)$ for a transition centered at $x_0$.

Preceding argument was *purely* *variational* – there's no PDE in sight. But it's related to method of matched asymptotic expansions. Our 1D fnl

$$\int_0^1 \varepsilon u_x^2 + \frac{1}{\varepsilon}(u^2-1)^2$$

has only the obvious local minima $u \equiv \pm 1$ (this was an exercise) but it has plenty of saddle pts, which solve

$$-2\varepsilon u_{xx} + 4(u^3-u) = 0 \quad \text{in} \quad (0,1)$$
$$u_x = 0 \quad \text{at endpts}$$

Low-index saddles can be understood by phase plane analysis - the "transitions" are more or less evenly spaced. One could try to represent the solution of pde by matched asymptotic expansion; the "inner expansion" would lead to the same transition profile we got above. (It's actually difficult to implement this rigorously, because the "outer expansion" involves only terms that are "exponentially small in $\varepsilon$.")

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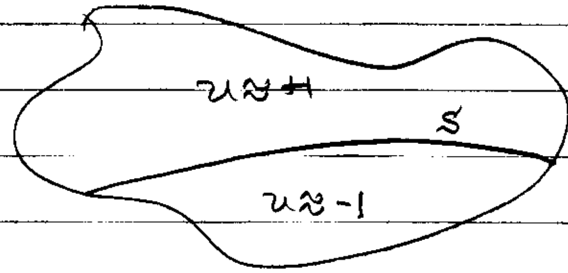
Multidim'l picture is similar - let's focus on 2D - except now "transitions" have more freedom (they're along curves, not at pts). Still, assertion of Modica-Mortola is that to get




we require the "energy in the transition layer near $S$" to be at least $8/3 \cdot \text{Length}(S)$.

Informal arg't is as before:

$$\int_\Omega \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon}(u^2-1)^2 \geq 2\int_\Omega |\nabla u|\,|u^2-1|$$

$$= 2\int_\Omega |\varphi'(u)\nabla u|$$

$$= 2\int_\Omega |\nabla \varphi(u)|$$

If $u_\varepsilon \to u_0$ (discont's as in picture) then $\varphi(u_\varepsilon) \to \varphi(u_0)$ (discont's across $S$) and

$$2\int |\nabla \varphi(u_0)| = 2[\varphi(1)-\varphi(-1)] \cdot \text{Length}(S)$$

since $\nabla\varphi(u_0)$ is "a $\delta$-function concentrated at $S$".

Moreover, argument shows that inequality is (almost) sharp if $u_\varepsilon(x) = \pm \tanh\left(\frac{d(x,S)}{\varepsilon}\right)$, where

$d(x,S)$ = signed distance to $S$.

More careful statements:

① if $v_\varepsilon \to v_0$ in $L^1$ as $\varepsilon \to 0$, then

$$\liminf F_\varepsilon(v_\varepsilon) \geq F_0(v_0)$$

② if $v_0 \in L^1$ then $\exists\, v_\varepsilon$ s.t. $v_\varepsilon \to v_0$ in $L^1$ +

$$F_\varepsilon(v_\varepsilon) \to F_0(v_0)$$

These constitute the definition of $F_\varepsilon$ $\Gamma$-converging to $F_0$ in the $L^1$ topology. Also we have

③ if $\{v_\varepsilon\}$ have $F_\varepsilon(v_\varepsilon)$ unif bdd, then $\{v_\varepsilon\}$ is compact in $L^1(\Omega)$

This is why it's natural to study $\Gamma$-conv. wrt the $L^1$ topology.

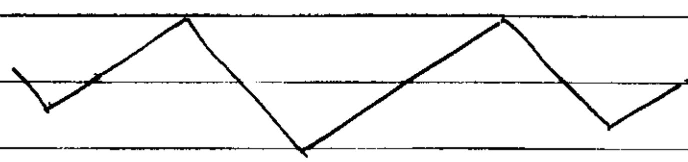
Our intuitive treatment should have made ① + ② plausible; we'll return to discuss them further (also ③) a bit later. But now let's return to the "consequences".

1st consequence (special case, in 1D): consider

$$E_\varepsilon(u) = \int_0^1 \underbrace{\varepsilon u_{xx}^2 + \frac{1}{\varepsilon}(u_x^2-1)^2}_{\substack{\text{Modica-Mortola} \\ \text{wrt to } u_x}} + \lambda u^2$$

In limit $\varepsilon \to 0$, it $\Gamma$-converges to

$$E_0(u) = \int_0^1 \lambda u^2 \, dx + \frac{8}{3} \cdot \#(\text{teeth}), \text{ if } u_x = \pm 1$$
(a "sawtooth").



In particular,

$$\min_u E_\varepsilon(u) \text{ converges to } \min E_0(u)$$

and if $u_\varepsilon$ minimizes $E_\varepsilon$, its limit pt(s) minimize $E_0$.

I've been a bit sloppy: since the anticipated limits are "sawtooth fns" the relevant topology $u_x \in L^1$, ie $W^{1,1}$. The lower order term $\lambda u^2$ is cpt in this topology, so it does not affect the

$\Gamma$-convergence (if $F_\varepsilon \xrightarrow[\Gamma]{} F_0$ as defined by ① + ②
Then $F_\varepsilon + G \xrightarrow[\Gamma]{} F_0 + G$.)

Claim: $\Gamma$-convergence $\Rightarrow$ convergence of minimizers, + minimizing value.

In fact: if $F_\varepsilon \xrightarrow[\Gamma]{} F_0$ then

$$\overline{\lim}\,(\min F_\varepsilon) \leq \min F_0 \quad \text{by ②}$$

but

$$\underline{\lim}\,(\min F_\varepsilon) \geq \min F_0 \quad \text{by ①}$$

So minimizing values converge. Therefore if $u_\varepsilon$ minimizes $F_\varepsilon$, $F_\varepsilon(u_\varepsilon) \to \min F_0$. Now suppose $u_\varepsilon \to u_0$. Then

$$\liminf F_\varepsilon(v_\varepsilon) \geq F_0(v_0). \quad \text{by ①}$$

So

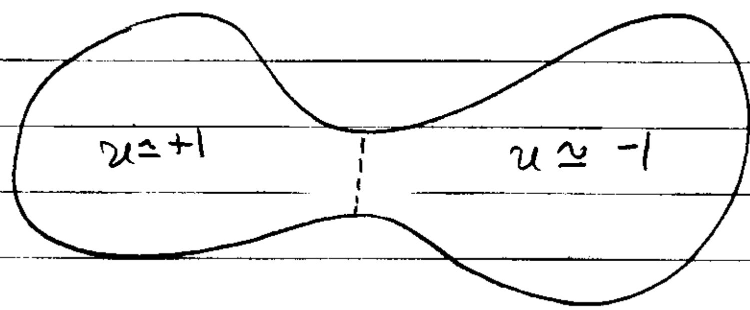
$$\min F_0 \geq F_0(v_0)$$

So $v_0$ must achieve $\min F_0$.

(Note: we used nothing special to Modica-Mortola here!).

Preceding consequence was "classical." The next is not so classical, but still easy (see the Kohn-Sternberg paper cited on pg 10.1)

Claim: suppose $\Omega$ has "a strictly convex neck"




Then

① The fn'l. $F_0 = \frac{8}{3} \cdot \mathrm{Per}_{\Omega}\{u=1\}$ of $u = \pm 1$

has an isolated $L^1$-local min $u_0$ a transection in the neck.

② $F_\varepsilon = \int_\Omega \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon}(u^2-1)^2$ has a local min $u_\varepsilon$ s.t. $u_\varepsilon \to u_0$ as $\varepsilon \to 0$.

Pf of ① is not trivial, but I won't do it (see the paper).

Pf of ② is similar to what we did before: consider

$$\min_{\|u-u_0\|_{L^1} \le \delta} F_\varepsilon(u),$$

If $u_\varepsilon$ is optimal, then either $\|u_\varepsilon - u_0\|_{L^1} = \delta$ or else the constraint was irrelevant.

<u>Claim</u>: for small $\varepsilon$ the constraint is irrelevant. In fact, by part ② of defn of $\Gamma$-conv,

$$\overline{\lim}\ F_\varepsilon(u_\varepsilon) \le F_0(u_0).$$

If for a seq $\varepsilon_j \to 0$ we had $\|u_{\varepsilon_j} - u_0\|_{L^1} = \delta$ then in limit $u_{\varepsilon_j} \to u_*$ we would have

$$\|u_* - u_0\|_{L^1} = \delta$$

but part ① of $\Gamma$-conv. says

$$\underline{\lim}\ F_\varepsilon(u_\varepsilon) \ge F_0(u_*)$$

whence $F_0(u_*) \le F_0(u_0)$. If $\delta$ is small but pos, this contradicts hypothesis that $u_0$ is an $L^1$-isolated local min.

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OK, so constraint was irrelevant; then $u_\varepsilon$ is $L^1$ local min of $F_\varepsilon$. Easy to see $u_\varepsilon$ must $\to u_0$

as $\varepsilon \to 0$: if not then there would be a seq $u_{\varepsilon_j}$ st

$$\|u_{\varepsilon_j} - u_0\|_{L^1} \geq \text{const} > 0$$

so any $L^1$ limit pt $u_{\varepsilon_j} \to u_*$ has

$$\|u_* - u_0\|_{L^1} \geq \text{const} > 0$$

and (as shown above) $F_0(u_*) \leq F_0(u_0)$.
Again, this contradicts hypoth that $u_0$ was an isolated local min.

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It remains to say a bit more about how properties ①, ②, ③ are proved for the Modica-Mortola example. Next set of notes...