Calculus of Variations, Lecture 1, 9/9/09

(See syllabus for semester plan, reserve list, etc.)

Today's topic: the "direct method of the calculus of variations"

Goal may be either
1) to minimize a functional that has intrinsic meaning (eg geodesics, optimal control, minimal surfaces, elasticity, etc)

or

2) to solve a pde by recognizing it as the EL eqn of a suitable functional

A simple case is probably familiar to most students: if $\Omega \subset \mathbb{R}^n$ is bdd domain, minimizer of var'l pbm

① $$\min_{u=\varphi \text{ at } \partial\Omega} \int_\Omega \tfrac{1}{2}|\nabla u|^2 + fu \, dx$$

solves
$$-\Delta u + f = 0 \quad \text{in } \Omega.$$
$$u = \varphi \quad \text{at } \partial\Omega$$

<u>Reminder of why</u>: if $E[u] = \int_\Omega \frac{1}{2}|\nabla u|^2 + fu \; dx$

$$1^{st} \text{ var'n of fnl} = \frac{d}{dt}\Big|_{t=0} E(u+tv)$$
$$= \int_\Omega \langle \nabla u, \nabla v\rangle + fv \; dx$$

So 1^{st} var'n $= 0$ at $u \iff u$ is a "weak soln" of $-\Delta u + f = 0$. (Test fns v must have $v|_{\partial\Omega} = 0$.)

To <u>see</u> u is a classical soln (eg u is $C^{2,\alpha}$ if f is C^α) need pde reg'y theory (var'l principle doesn't easily give regularity).

Soln is unique due to <u>strict</u> <u>convexity</u> of E: if u solves EL eqn (ie it is a critical pt of E) + $u|_{\partial\Omega} = 0$ then for any w st $w|_{\partial\Omega} = \varphi$,

$$E[w] = E[u] + 1^{st} \text{ var'n at } u \text{ in dirn } w-u$$
$$+ \; 2^{nd} \text{ var'n term that's pos}$$

which simplifies in this quadratic setting to

$$\int_\Omega \frac{1}{2}|\nabla w|^2 + fw = \int_\Omega \frac{1}{2}|\nabla u|^2 + fu + \int_\Omega \langle \nabla u, \nabla(w-u)\rangle + f(w-u)$$
$$+ \int_\Omega \frac{1}{2}|\nabla u - \nabla w|^2 \; dx$$

Middle term on RHS vanishes if u is crit pt of E. So $E[w] \geq E[u]$ with equality only when $\int |\nabla u - \nabla w|^2 = 0$ i.e. $u = w$ (remembering that $u - w = 0$ at $\partial\Omega$, from the bc).

Existence of weak soln is most often proved by Lax-Milgram lemma, but minzn of E provides alternative route. Key advantage: it generalizes to nonlinear pde (nonquadratic var'l prbms). We'll discuss var'l proof of existence at end of this lecture.

Direct method looks easy, but there are subtleties. Let's identify them by giving some examples of things that can go wrong.

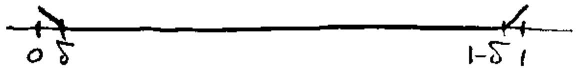
(A) In view of preceding discn abt var'l prin for $-\Delta u + f = 0$ with Dir bc, one might be tempted to solve this eqn with Neumann bc (say, $\partial u/\partial n = g$ at $\partial\Omega$) by

(2) $$\min_{\frac{\partial u}{\partial n} = g \text{ at } \partial\Omega} \int_\Omega \frac{1}{2}|\nabla u|^2 + f u \, dx \qquad \boxed{\text{WRONG}}$$

but that <u>doesn't work</u>. Consider for example the 1D <u>case</u> with $g = 1$ and $f = 0$

$$\min_{\substack{u_x(0)=-1 \\ u_x(1)=+1}} \int_0^1 \frac{1}{2} u_x^2 \; dx$$

The min value is <u>zero</u>; a min sequence looks like



Limit of this min seq. is constant, so $u_{xx} = 0$ but <u>bc is lost</u>. (Exercise: what happens when $f \neq 0$? What about dim $n \geq 2$?)

Correct var'l prin for Neumann pblm recovers bc as part of EL eqn

(3) $$\min_u \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + f u\right) dx - \int_{\partial\Omega} u g \, ds$$

since crit pt now satisfies

$$\int_\Omega \langle \nabla u, \nabla v \rangle + f v \, dx - \int_{\partial\Omega} g v \, ds$$

for all $v \in H^1(\Omega)$. If u is smooth enough this gives

$$\int_{\partial\Omega} \left(\frac{\partial u}{\partial n} - g\right) v \, ds + \int_{\Omega} (-\Delta u + f) v \, dx = 0$$

Since v is unrestricted at $\partial\Omega$, we get both pde + Neumann bc.

(B) We must take care that min is not $-\infty$. For Neumann pblm discussed above, we expect something to go wrong if f + g don't satisfy consistency condn

$$\int_{\partial\Omega} g = \int_{\Omega} f$$

And indeed, if consistency fails then min $= -\infty$, as we see by taking $u = c =$ constant (letting $c \to \infty$ or $-\infty$).

Pf that min is not $-\infty$ when data is consistent uses the inequality

$$\int_{\Omega} |\nabla u|^2 \geq C \int_{\Omega} |u|^2 \quad \text{if} \quad \int_{\Omega} u = 0.$$

(Note: best C is 1st nonzero eigenvalue of

Neumann Laplacian on Ω. Of course it depends on Ω.) We also need inequality

$$\|u\|_{L^2(\partial\Omega)} \le C \|u\|_{H^1(\Omega)}$$

(ie: the map $H^1(\Omega) \to L^2(\partial\Omega)$ taking a fn u to its "boundary trace" is cont's; actually we need this implicitly simply to know that ③ makes sense). Consistent data $\Rightarrow$ var'l prin is insensitive to replacement of u by $u + \text{const}$ $\Rightarrow$ may restrict attn to u s.t. $\int_\Omega u = 0$. For such u, preceding ineqs easily give

$$\int_\Omega f u \, dx - \int_{\partial\Omega} u g \, ds \le C \|u\|_{H^1(\Omega)}$$
$$\le C' \left(\int |\nabla u|^2 \right)^{1/2}.$$

(assuming $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$); so var'l prin is bdd below by

$$\min_z \tfrac{1}{2} z^2 - C' z$$

which is finite.

(C) In nonlinear setting, even a Dir bc can be "lost" in the course of minimization, if energy behaves as $|\nabla u|$ near ∞. Consider, for example,

$$\text{(4)} \quad \min_{u=\varphi \text{ at } \partial\Omega} \int_\Omega \left(1+|\nabla u|^2\right)^{1/2} dx.$$

i.e.

min (surface area of graph of u, for given bdry data)

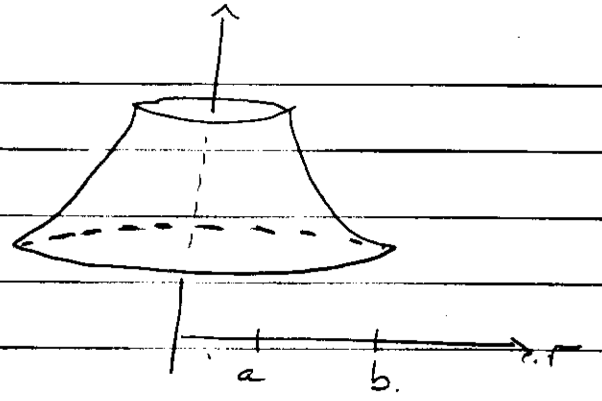
When Ω is strictly convex the min is achieved, but proving this is nontrivial. More interesting today: when Ω is nonconvex the min might not be achieved. A simple example is the case when

$$\Omega = \text{annulus } \{a < r < b\} \subset \mathbb{R}^2$$

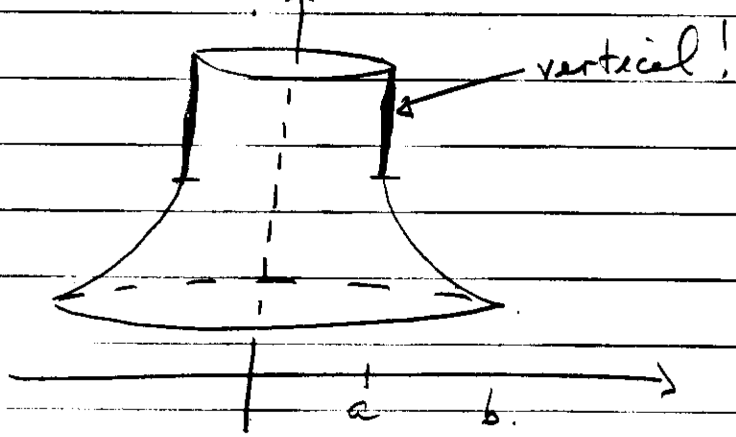
and bc is

$$\varphi = \begin{cases} A & \text{at } r=a \\ B & r=b \end{cases}$$

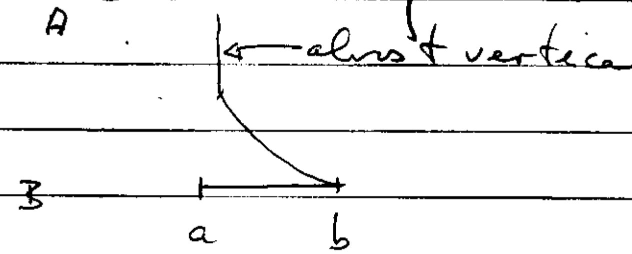
with $A-B$ suffly large. In fact, if $A>B$ but $A-B$ is not too large, soln is achieved



but when A-B is large enough the area-minimizing surface has a vertical piece at inner edge



In latter case min sep. looks like



(Obviously, there's a problem with the fnl. It can be fixed by considering instead the "relaxed problem

$$⑤ \quad \min \int_{\Omega} (1+|\nabla u|^2)^{1/2}\, dx + \int_{\partial\Omega} |u-\varphi|\, ds$$

which correctly accounts for vertical pieces at

The bdry. Key pts: a) relaxed problem achieves its min

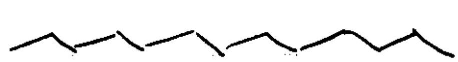
b) min seq of original pbm achieves min of relaxed pbm

Proving these assertions for 4 + 5 is not trivial. Suggestion: try doing it in the radial case.)

For more details, including formulas assoc to the pictures above, see pp 47-48 of the book by Buttazzo, Giaquinta, + Hildebrandt.

(D) If fn'l is nonconvex, then a min sequence can easily develop oscillations. Its (weak) limit will not achieve the min. For example:

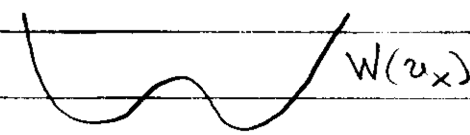
$$\min_u \int_0^1 (u_x^2-1)^2 + u^2 = 0$$

but min is not achieved. Min sequence looks like 

(E) In nonconvex case, we can get a "local" min, but it's important how we interpret the word "local." For example: consider

$$\min \int_0^1 (u_x^2-1)^2 \, dx \qquad \text{subject to } u(0)=u_0, \quad u(1)=u_1$$

+ write $W(u_x) = (u_x^2-1)^2$

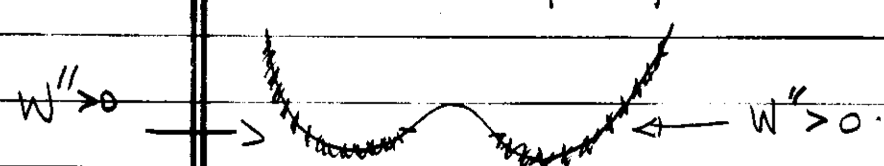


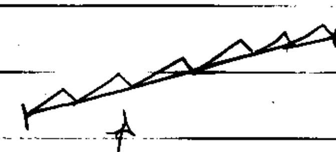
Then EL eqn is $(W'(u_x))_x = 0$. Linear fn u_* is clearly a soln. If linear fn has $W''(u_{*x}) > 0$ then 2nd varn is positive, since

$$\frac{d^2}{dt^2}\Big|_{t=0} \int_0^1 W(u_{*x} + t v_x) = \frac{d}{dt}\Big|_{t=0} \int_0^1 W'(u_{*x} + t v_x)\, v_x$$

$$= \int_0^1 W''(u_{*x})\, v_x^2$$

which is strictly positive if $v(0) = v(1) = 0$. One can show that u_* achieves a <u>local min in C^1</u>. However it is <u>not</u> a local min in L^∞, since an oscillating fn does better (if $|u_{*x}|<1$).





u_* st $0 < u_{*x} < 1$ + $W''(u_{*x}) > 0$, and an osc fn that does better.

(F) Sometimes it matters what space we minimize over, even when we seek absolute rather than local minimizer. Two examples:

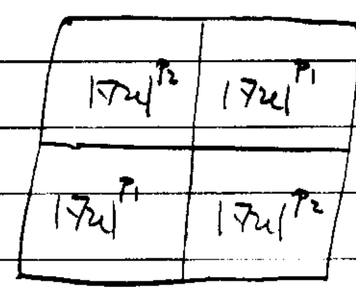
a) $$\min_{\substack{u(0)=0\\ u(1)=1}} \int_0^1 (u^3-x)^2\, u_x^6 \, dx$$

is clearly minimized by $u(x) = x^{1/3}$, which achieves value 0. However if u is restricted to the class of Lipschitz cont. fns on $[0,1]$ then the min is bdd away from zero.

See 3.4.3 of Buttazzo - Giaquinta - Hildebrandt for a proof. (This example is due to Manià.)

b) let $$p(x) = \begin{cases} p_1 & \text{in quadrants } 1,3 \\ p_2 & \text{in quadrants } 2,4 \end{cases} \quad \text{in } \mathbb{R}^2.$$

+ consider $$\min_{\substack{u=\varphi \\ \cancel{\text{at bdry}}}} \int_{[0,1]\times[0,1]} |\nabla u|^{p(x)}\, dx$$



If $p_1 < p_2$, it's clear that

$$\min \text{ over } u \in W^{1,p_1} \le \min \text{ over } u \in W^{1,p_2}$$

since $\left(\int_\Omega |\nabla u|^{p_1}\right)^{1/p_1} \leq C\left(\int_\Omega |\nabla u|^{p_2}\right)^{1/p_2}$ when $p_1 < p_2$ (by Holder's ineq, provided Ω has finite volume). In fact the inequality is strict: if $p_1 < p_2$ then

$$\text{min value over } u \in W^{1,p_1} < \text{min value over } u \in W^{1,p_2}$$

strict!

(This example is due to Zhikov. See eg "On Lavrentiev's Phenomenon" by VV Zhikov, Russian J Math Physics vol 3, no 2, 249-269.)

This phenomenon (that the min value depends on the choice of fn space) is called the "Lavrentiev phenomenon". It can arise when the minimizer (in the larger space) is singular, + therefore cannot be approximated (with about the same value for the "energy") in the smaller space.

When such behavior occurs, we should also ask: does the minimizer satisfy the EL eqn? (The issue: is $t \to E[u+tv]$ differentiable at $t=0$?)

OK, after so many subtleties one might ask: how can we ever justify using the direct method?

Here let's discuss just the simple case

$$\min_{u=\varphi \text{ at } \partial\Omega} \int_\Omega W(\nabla u) + f u \; dx$$

where W is convex with

$$C_1(|\xi|^p - 1) \le W(\xi) \le C_2(\xi^p + 1) \quad \text{for some } p>1.$$

(For a more general treatment, eg for $\int W(x, u, \nabla u)$ where W is convex in ∇u but merely conts in u + meas in x, see Chapter 4 of Jost/Li-Jost.)

step 0: To be sure the functional is well-defined we must impose some restrictions on φ + f.

For φ, let's just assume $\exists\, \Phi \in W^{1,p}(\Omega)$ st $\Phi|_{\partial\Omega} = \varphi$; then our problem can be written as

$$\min_{\tilde{u} = u - \Phi \in W_0^{1,p}(\Omega)} \int_\Omega W(\nabla\Phi + \nabla\tilde{u}) + f\cdot(\Phi + \tilde{u}) \; dx$$

where $W_0^{1,p}(\Omega)$ = closure of cptly sptd fns in $W^{1,p}$ norm

$= \{ u \in W^{1,p}(\Omega) \text{ st } u|_{\partial\Omega} = 0 \}.$

For f, we need

$$\int f u \le C \|u\|_{W^{1,p}(\Omega)}$$

ie the term $\int f u$ must be a cont's function of $u \in W^{1,p}(\Omega)$. It's sufficient that $f \in L^q(\Omega)$, $\frac{1}{q} + \frac{1}{p} = 1$, since then

$$\int_\Omega f u \le C\|f\|_{L^q} \|u\|_{L^p} \le C\|f\|_{L^q} \|u\|_{W^{1,p}}$$

Step 1: Show the functional is bounded below on $W^{1,p}(\Omega) \cap \{u : u = \varphi \text{ at } \partial\Omega\}$. By hypothesis

$$\int W(\nabla u) \ge C \int |\nabla u|^p - \text{const}$$

But $u = \Phi + \tilde{u}$ with $\tilde{u} \in W_0^{1,p}(\Omega)$. Using the Poincaré-type ineq

$$\int_\Omega |\tilde{u}|^p \le C \int_\Omega |\nabla \tilde{u}|^p \quad \text{for all } \tilde{u} \in W_0^{1,p}(\Omega)$$

we get

$$\begin{aligned} \|u\|_{W^{1,p}} &\le \|\Phi\|_{W^{1,p}} + \|\tilde{u}\|_{W^{1,p}} \\ &\le C\left(\int |\nabla \tilde{u}|^p\right)^{1/p} + \text{const} \end{aligned}$$

$$\leq C\left(\int |\nabla u|^p\right)^{1/p} + \text{new constant}$$

using the triangle inequality. So

$$\int W(\nabla u) \geq C \int |\nabla u|^p - \text{constant}$$

(of course my constants "C" change from line to line).

Since $\min_{z \geq 0} C_1 z^p - C_2 z$ is finite,

we see that the functional is bdd below.

Moreover: for any μ, the set

$$\left\{u : \int_\Omega W(\nabla u) + fu \, dx \leq \mu, \ u = \varphi \text{ at } \partial\Omega\right\}$$

is a bounded set in the $W^{1,p}(\Omega)$ norm.
(This is an easy consequence of the preceding ests.)

Step 2: From step 1 we can take a min seq $\{u_k\}_{k=1}^\infty$ and any such sequence stays bdd in $W^{1,p}$ norm.

Since $p>1$, unit ball of $W^{1,p}$ is cpt under weak convergence. So $\exists$ subseq st $u_{k_j} \rightharpoonup u_*$.

Want to show u_* is a minimizer. Key pt: a closed, convex subset of $W^{1,p}$ is closed under weak convergence. Apply this to

$$\{u \in W^{1,p}(\Omega) \text{ st } u=\varphi \text{ at } \partial\Omega \text{ and } \int_\Omega W(\nabla u) + f\cdot u \, dx \leq m \}$$

for any $m >$ min value (this set is convex because W is convex) to see that

$$\int W(\nabla u_*) + f u_* \leq \liminf_{k_j} \int W(\nabla u_{k_j}) + f u_{k_j}$$

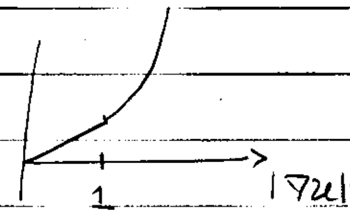
(ie our fnl is lower semicontinuous). Since u_{k_j} was a minimizing sequence, u_* must achieve the min.

$$\int_\Omega W(\nabla u_*) + f u_* = \inf_{\substack{u \in W^{1,p} \\ u=\varphi \text{ at } \partial\Omega}} \int_\Omega W(\nabla u) + f u$$

Rmk: Proving existence this way may seem abstract. But this is also the basis of a very practical numerical scheme: to find the minimizer numerically by finite element method, one simply minimizes the functional in a piecewise-polynomial subspace of $W^{1,p}(\Omega)$.

<u>Does minimizer satisfy the EL eqn?</u> With no further hypotheses, EL eqn is very singular; eg if

$$W(\nabla u) = \begin{cases} 2|\nabla u| & |\nabla u| \le 1 \\ 1+|\nabla u|^2 & |\nabla u| \ge 1 \end{cases}$$



Then W is C^1 except at 0, but EL eqn

$$-\operatorname{div}\left(\frac{\partial W}{\partial \nabla u}\right) + f = 0$$

is undefined at $\nabla u = 0$. For problems like this we need convex duality as a substitute for EL eqn.

But if W is suffly diff/ble then there's no problem: suppose $\xi \to W(\xi)$ is differentiable and

$$\left|\frac{\partial W}{\partial \xi}\right| \le C\left(|\xi|^{p-1} + 1\right)$$

Then the usual derivation of the EL eqn is justified (we can differentiate under the integral) since $u, v \in W^{1,p} \Rightarrow$

$$\int_\Omega \frac{d}{dt} W(\nabla u + t\nabla v) = \int_\Omega \left\langle \frac{\partial W}{\partial \nabla u}(\nabla u + t\nabla v), \nabla v \right\rangle$$

$$\le \left(\int_\Omega \left|\frac{\partial W}{\partial \nabla u}(\nabla u + t\nabla v)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \cdot \left(\int |\nabla v|^p\right)^{\frac{1}{p}}$$

is controlled (indep of t) by the $W^{1,p}$ norms of $u+v$.

Suggested reading related to Lecture 1:

- Sections 1.3 of Buttazzo - Giaquinta - Hildebrandt ("some classical problems") and Section 1.1 of that book ("the Euler equation and other necessary conditions for optimality") focus on 1D variational problems, thereby avoiding subtleties of functional analysis and getting quickly to interesting results + examples.

- Chapter 4 of Jost/Jost-Li covers existence by the direct method, for problems somewhat more general than those discussed above.

Comment: the hypothesis that $W(\nabla u)$ be a convex fn of ∇u is natural when u is scalar-valued, but not when u is vector-valued. In fact, in vector-valued setting $\int W(\nabla u)$ can be lower semicontinuous even if W is not convex. We'll return to this later.

Suggested exercises related to Lecture 1:

1) I gave two examples where the "direct method" fails because the bndry condition is lost in the limit (example A and example C). Which steps in our "proof" of the direct method fail in each case, and why?

2) I mentioned at the end that we can "solve" a variational problem numerically by minimizing the functional over a finite-dimensional subspace (for example, piecewise linear functions on a particular triangulation). Justify this for the special case

$$\Delta u = f \text{ in } \Omega.$$
$$u = 0 \text{ at } \partial\Omega$$

by showing that for any subspace $V \subset W_0^{1,2}(\Omega)$, the solution w_* of

$$\min_{w \in V} \int_\Omega \tfrac{1}{2}|\nabla w|^2 + wf$$

satisfies

$$\int_\Omega |\nabla u - \nabla w_*|^2 = \min_{w \in V} \int |\nabla u - \nabla w|^2.$$

(The RHS is easy to estimate, using the smoothness of u.) Food for thought: how can this type of result be generalized to $\int W(\nabla u) + fu$ when W is convex but not quadratic?

3) Does the variational problem

$$\min_{\substack{u=0 \\ \text{at } \partial\Omega}} \int_\Omega \tfrac{1}{2}|\nabla u|^2 + fu \; dx$$

achieve its minimum when $f = \delta_{x_0}$ for some $x_0 \in \Omega$ (so $\int_\Omega fu = u(x_0)$)? Hint: dim $n \geq 2$ is different from dim 1.

4) What pde and boundary condition holds at critical pts of

$$\int_\Omega \tfrac{1}{2}|\nabla u|^2 + u \cdot f \; dx + \beta \int_{\partial\Omega} |u|^2 \quad ?$$

5) Suppose W is strictly convex and C^2, and u is a classical solution of

$$\operatorname{div}\left(\frac{\partial W}{\partial \nabla u}(\nabla u)\right) = 0 \quad \text{in } \Omega$$

with bdry condition $u = \varphi$ at $\partial\Omega$. Show that u achieves

$$\min_{u=\varphi} \int W(\nabla u)$$

and that the unique (classical) minimizer.

6) Consider $W(u_x) = (u_x^2-1)^2$, and suppose

$$W''(b-a) > 0.$$

Show that the linear function $u_*(x) = a + (b-a)x$ is a C^1-local minimizer of

$$\int_0^1 W(u_x)\,dx \quad \text{subject to } u(0)=a,\ u(1)=b$$

in the sense that for any $v \in C^1(0,1)$ with the same bdry conds + $\|v-u\|_{C^1}$ suff'tly small we have

$$\int_0^1 W(v_x)\,dx > \int_0^1 W(u_{*x})\,dx.$$

(Hint: start by showing that the function

$$t \to \int_0^1 W(u_{*x} + t(v_x - u_{*x}))\,dx$$

is convex.)