Linear Algebra I Practice Final Examination

No books or calculators are permitted in this examination, however you are allowed three page of
notes (8.5 × 11, both sides). Be sure to show intermediate steps and carefully explain your answers
to ensure maximum credit. The duration of the examination is 105 minutes.

(1) [15 points] Consider the following matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

Calculate the eigenvalues and corresponding eigenvectors. Is the matrix diagonalizable? If so find $P$ and $D$ such that $A = PDP^{-1}$ where $D$ is diagonal. If not carefully explain why it is not possible.

(2) [15 points] Consider the matrix

$$B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Do there exist matrices $P$ and $D$ such that $B = PDP^T$ where $P^{-1} = P^T$ and $D$ is diagonal? Why? If these matrices exist then write down a possible $P$ and the corresponding $D$.

(3) [15 points] Suppose a matrix $A$ satisfies $A^2 = 0$ where 0 is the matrix consisting of all zeros. Show that det $(A) = 0$. What are the possible eigenvalues for $A$?

(4) [15 points] Consider points in the $(x,y)$ plane. Suppose we have the following four data points

$$(2, 2), (1, 1), (-1, -2), (-2, -3)$$

Fit the least squares straight line to these points. Find the root mean square error in $y$ per
data point for this straight line fit.

(5) [15 points] Consider the following three vectors in $\mathbb{R}^3$

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

Are these vectors linearly independent? Which are orthogonal? Starting with any orthogonal
vectors use the Gram-Schmidt process to produce a new set of three orthogonal vectors.

(6) [15 points] Suppose we have an orthogonal $2 \times 2$ matrix $P$ i.e. $P^T = P^{-1}$. Consider the column
vectors of this matrix. What length are they? What angle is there between the two column
vectors? Using this information and simple geometry write down general expressions for both
column vectors in terms of the angle $\theta$ that the first column vector makes with the x axis i.e.
the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Hints: The vector tips lie on a circle. There are two solutions.
(7) [15 points] Consider the following quadratic form

\[ Q(x, y, z) = x^2 + y^2 + z^2 + 2xy + 2yz \]

Write this in matrix form i.e. find a symmetric matrix \( A \) such that

\[ Q = \begin{pmatrix} x \\ y \\ z \end{pmatrix}^T A \begin{pmatrix} x \\ y \\ z \end{pmatrix} \]

Suppose that in addition

\[ x^2 + y^2 + z^2 = 1 \]

Is \( Q \) positive definite? What is the minimum value that \( Q \) can take? What values of \( x, y, z \) achieve this minimum?