Linear Algebra Lecture 1

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January 2001

1 Basic Vector Facts

Vectors have direction and magnitude. Examples are legion: Forces and Velocities are the best known.

Vectors obey geometric laws for addition (the parallelegram law) and multiplication by a scalar.

Vectors can be described algebraically with analytical geometry. This involves $n'tuples$.

Such objects can be shown to satisfy a number of rules:

1. $\forall \overrightarrow{x}, \overrightarrow{y}$ we have $\overrightarrow{x} + \overrightarrow{y} = \overrightarrow{y} + \overrightarrow{x}$ (abelian or commutative property.
2. $\forall \overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}$ we have $(\overrightarrow{x} + \overrightarrow{y}) + \overrightarrow{z} = \overrightarrow{x} + (\overrightarrow{y} + \overrightarrow{z})$ (associative property).
3. $\forall \overrightarrow{x}$ $\exists$ a vector $\overrightarrow{0}$ which satisfies $\overrightarrow{x} + \overrightarrow{0} = \overrightarrow{x}$ (additive identity).
4. For any vector $\overrightarrow{x}$ $\exists$ another vector $\overrightarrow{y}$ such that $\overrightarrow{x} + \overrightarrow{y} = \overrightarrow{0}$ (additive inverse)
5. $\forall \overrightarrow{x}$ we have $(1)\overrightarrow{x} = \overrightarrow{x}$ (scalar multiplication identity).

6. $\forall a, b$ which are numbers and $\forall \overrightarrow{x}$ we have $(ab)\overrightarrow{x} = a(b\overrightarrow{x})$
7. $\forall a$ and $\forall \overrightarrow{x}, \overrightarrow{y}$ we have $a(\overrightarrow{x} + \overrightarrow{y}) = a\overrightarrow{x} + a\overrightarrow{y}$
8. $\forall a, b$ which are numbers and $\forall \overrightarrow{x}$ we have $(a+b)\overrightarrow{x} = a\overrightarrow{x} + b\overrightarrow{x}$

Lines and Planes have can be described by families of vectors.

Vector spaces V are formally defined over a field F and if $\overrightarrow{x}, \overrightarrow{y} \in V$ then there is a unique element $\overrightarrow{x} + \overrightarrow{y} \in V$ and a unique element $a\overrightarrow{x} \in V$ where a is a genaral element from the field F. In addition the elements of V satisfy the 8 rules above.

Proposition 1.1: Cancellation Law: $\forall \overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z} \in V$ we have $\overrightarrow{x} + \overrightarrow{z} = \overrightarrow{y} + \overrightarrow{z} \Rightarrow \overrightarrow{x} = \overrightarrow{y}$

Corollary: The vector $\overrightarrow{0}$ is unique

Corollary: The additive inverse is unique.

Proposition 1.2:

1. $0\overrightarrow{x} = \overrightarrow{0}$ $\forall \overrightarrow{x} \in V$
2. $(-a)\overrightarrow{x} = -(a\overrightarrow{x}) = a(-\overrightarrow{x})$ $\forall \overrightarrow{x} \in V$ and $\forall a \in F$
3. $a\overrightarrow{0} = \overrightarrow{0}$ $\forall a \in F$.

2 Subspaces

Subset W of V is called a subspace if it is a vector space over F and under the addition and scalar multiplication defined on V.

Only a subset of axioms need be checked since axioms 1,2 and 5-8 hold for all elements of V and hence W. Need to check therefore only:

1. $\forall \overrightarrow{x}, \overrightarrow{y} \in W \ \overrightarrow{x} + \overrightarrow{y} \in W$ (Closure under addition).
2. $\forall a \in F$ and $\overrightarrow{x} \in W$ we have $a\overrightarrow{x} \in W$ (Closure under scalar muliplication)
3. W has a zero vector (must be the same as V of course).
4. The additive inverse of all elements of W must lie in W.

Theorem 1.3: Let V be a vector space and $W \subset V$ then W is a subspace of V under the operations defined on V $i.f.f.$

1. $\overrightarrow{0} \in W$
2. If $\overrightarrow{x}, \overrightarrow{y} \in W$ then $\overrightarrow{x} + \overrightarrow{y} \in W$
3. If $a \in F$ and $\overrightarrow{x} \in W$ then $a\overrightarrow{x} \in W$

Theorem 1.4: Any intersection of subspaces of a vector space is also a subspace.

The union of subspaces is not a subspace in general because it is not closed under addition.

Theorem 1.5: If W_1 and W_2 are subspaces of V then $W_1 \cup W_2$ is a subspace $i.f.f.$ $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

Definition: The sum of two subsets S_1 and S_2 of V is the set $S_1 + S_2 = \{x + y : \forall x \in S_1 \, and \, \forall y \in S_2\}$

Theorem 1.6: If W_1 and W_2 are subspaces of V then

1. $W_1 + W_2$ is a subspace of V containing both $W_1$and W_2.
2. Any subspace of V containing both W_1 and W_2 must also contain $W_1 + W_2$.

Definition: A vector space V is the direct product of two subspaces V_1 and V_2 if

1. $V_1 \bigcap V_2 = \emptyset$
2. $V_1 + V_2 = V$

One writes $V = V_1 \bigoplus V_2$.