A Comparison of the Influence of Additive and Multiplicative Stochastic Forcing on a Coupled Model of ENSO

Cristina L. Perez
Department of Applied Physics and Applied Mathematics, Columbia University, New York, New York

Andrew M. Moore
Program in the Atmospheric and Oceanic Sciences, and Cooperative Institute for Research in Environmental Sciences, University of Colorado, Boulder, Colorado

Javier Zavala-Garay
Rosenstiel School of Marine and Atmospheric Science, University of Miami, Miami, Florida

Richard Kleeman
Courant Institute of Mathematical Sciences, New York University, New York, New York

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ABSTRACT

A currently popular idea is that El Niño–Southern Oscillation (ENSO) can be viewed as a linear deterministic system forced by noise representing processes with periods shorter than ENSO. Also, there is observational evidence to suggest that the Madden–Julian oscillation (MJO) acts to trigger and/or amplify the warm phase of ENSO in this way. The feedback of the slower process, ENSO, to higher-frequency atmospheric phenomena, of which a large part of the variability in the intraseasonal band is due to the MJO, has received little attention. This paper considers the hypothesis that the probability of an El Niño event is modified by high MJO activity and that, in turn, the MJO is regulated by ENSO activity. If this is indeed the case, then viewing ENSO as a low-frequency oscillation forced by additive stochastic noise would not present a complete picture.

This paper tests the above hypothesis using a stochastically forced intermediate coupled model by allowing ENSO to directly influence the stochastic forcing. The model response to a variety of stochastic forcing types is found to be sensitive to the type of forcing applied. When the model is operated beyond its intrinsic Hopf bifurcation, its probability distribution function (PDF) is fundamentally altered when the stochastic forcing is changed from additive to multiplicative. The model integration period also influences the shape of the PDF, which is also compared to the PDF derived from observations. It is found that multiplicative stochastic forcing reproduces some measures of the observations better than the additive stochastic forcing.

1. Introduction

Many studies suggest that atmospheric phenomena can act to stochastically force the coupled ocean–atmosphere system in the tropical Pacific and produce aperiodic variability of El Niño–Southern Oscillation (ENSO; e.g., Alexander and Penland 1996; Blanke et al. 1997; Eckert and Latif 1997; Kleeman and Moore 1997; Lau and Chan 1986, 1988; Moore and Kleeman 1999a; Penland 1996; Penland and Matrasova 1994; Penland and Sardeshmukh 1995; Schopf and Suarez 1988; Wainer and Webster 1996; Zavala-Garay et al. 2003). In this work we consider the interannual variability of ENSO as the low-frequency limit of the system, and any atmospheric phenomenon that exhibits higher-frequency variability than interannual as potentially stochastic forcing for the coupled ocean–
atmosphere system. One might think of weather events as the fast time-scale phenomenon or a process operating on intraseasonal time scales like the Madden-Julian oscillation (MJO), which is the strongest source of intraseasonal variability in the Tropics (Madden and Julian 1994). This establishes a frequency separation of roughly 1–2 yr⁻¹ between ENSO and the dominant source of stochastic forcing, intraseasonal variability. On ENSO time scales the latter can therefore be viewed as a stochastic process.

Theory suggests that the response of a system to stochastic forcing will be different depending upon whether the forcing is additive or multiplicative (Horsthemke and Lefever 1984). Additive stochastic forcing has a predetermined distribution, while multiplicative forcing depends upon the deterministic component of the system, also referred to as the “environment” (Horsthemke and Lefever 1984; Perez 2003). In the multiplicative case the stochastic forcing is a function of the environment and is therefore constantly evolving with the environment. Multiplicative forcing therefore has the ability to fundamentally change the dynamical characteristics and properties of the system. The physical system considered here consists of the low-frequency coupled ocean–atmosphere modes, the deterministic environment referred to above, and stochastic forcing of these modes by faster time-scale variability such as the MJO. The role played by multiplicative stochastic forcing in other geophysical systems has also been studied in relation to turbulence in large-scale rotating flows (DelSole 2001), atmospheric Rossby waves (Sardeshmukh et al. 2001), midlatitude barotropic flow over topography (Sura 2002), synoptic variability of midlatitude sea surface winds (Sura 2003), and climate drift in general circulation models (Sardeshmukh et al. 2003).

Previous studies of the influence of stochastic forcing on ENSO have focused on additive noise to characterize the effects of higher-frequency atmospheric phenomena on the slowly evolving coupled ocean–atmosphere modes. However, while the stochastic components of the coupled system can force the low-frequency coupled modes of interannual variability, the stochastic forcing can also be modulated by the low-frequency environment.

To illustrate, consider the work of Delcroix et al. (1993), who describe some of the effects of the westerly wind events upon the equatorial Pacific Ocean. In their study using shipboard and mooring instruments at 165°E longitude, they observed that during strong westerly wind events (≥4 m s⁻¹), the upper 50 m of the ocean becomes isothermal and the sea surface temperature (SST) cools. Delcroix et al. (1993) also observe that the ocean currents respond to the westerly wind bursts (WWBs) in that surface currents flowing westward at the equator decelerate in response to the WWBs almost immediately (within 2–3 days) and have on occasion reversed. Such observations suggest that WWBs represent an additive stochastic process.

However, other observations suggest that the MJO and WWBs appear to be modulated by ENSO, supporting the idea that the stochastic effects of intraseasonal variability are of a multiplicative nature. For example, L Lukas (1987) showed that changes in the intensity and location of convective activity are often associated with changes in the location and size of the warm pool, particularly on ENSO time scales. McPhaden and Picaut (1990) also suggest that through the ENSO-induced displacements of the western equatorial warm pool, MJO-related deep convection can migrate farther to the east (east of the date line even). Keen (1987) observed that WWBs exhibit interannual variability that is locked to the Southern Oscillation, with low values of the Southern Oscillation index (SOI) coinciding with peaks in the frequency of WWBs, and with WWBs generally occurring farther east. Results from an investigation by Harrison and Vecchi (1997) of a 10-yr dataset on the interaction between westerly wind events (WWEs) and ENSO indicate that the frequency and intensity of WWEs in the west-central equatorial Pacific increases when SOI is at its lowest. Negative correlation of this particular class of WWE with the SOI indicates a positive correlation with El Niño conditions, suggesting a relationship between the two, but in a linear sense and without establishing causation. Lau and Sui (1997) describe how short-term fluctuations (weeks to months) in local SST can affect the MJO and how MJO-associated convection, in turn, affects the upper ocean. They support their conclusions with the Tropical Ocean Global Atmosphere (TOGA) Comprehensive Ocean–Atmosphere Data Set (COARE) observations, which show that SST variations of more than 1°C are possible in response to MJO-induced changes in cloud cover. Their work emphasizes the complex nature of the feedback between ocean and atmosphere in the western Pacific warm pool on a hierarchy of time scales. More recently, Yu et al. (2003) performed a case study using the observed sea level pressure gradient between 130° and 160°E to study the potential effect of ENSO on the genesis of WWBs.

An example of a specific mechanism by which ENSO may be influenced by multiplicative MJO-related stochastic forcing was proposed by Zhang (2001), who showed using observations that intraseasonal SST perturbations in the eastern Pacific are associated with surface westerly wind variability in the western and central
Pacific due to the MJO. These SST anomalies have amplitudes in excess of 0.5°C, are zonally elongated, are generally symmetric about the equator, and are apparently not due to local atmospheric forcing but are caused by remote MJO forcing, which initiates downwelling equatorial ocean Kelvin waves. Zhang (2001) hypothesized a simple feedback loop between MJO and ENSO whereby intraseasonal changes in the zonal distribution of SST can modify the intensity of the easterly trade winds intraseasonally. If the trade winds are weakened, then equatorial upwelling will weaken in response, giving the warm pool an opportunity to extend farther eastward. The larger zonal extent of the warm pool offers the MJO a warm lower boundary with larger eastward zonal extent. Hence, westerlies associated with the MJO may travel farther into the central Pacific. Such westerly wind anomalies can initiate downwelling Kelvin waves farther east, thus reinforcing the Kelvin waves excited in the west. A series of such Kelvin waves will deepen the thermocline and cause warming of SST in the east.

There are also model studies that support the idea that a multiplicative approach is a physically sound basis for modeling the coupled ocean–atmosphere system in the tropical Pacific. Wang and Xie (1998) identify an MJO-like mode in a theoretical coupled model of the tropical ocean and atmosphere that is affected by the model ocean mixed layer thermodynamics. Their work suggests that the upper-ocean actively works to sustain the MJO through mechanisms that destabilize atmospheric moist Kelvin waves and that slow down atmospheric wave phase propagation to set up the 40–50-day time scale that is characteristic of the MJO.

In this paper, we explore the influence that observation-derived stochastic wind stress forcing, both additive and multiplicative, has on ENSO using an intermediate coupled model. In section 2 we present the method by which we generate the noise forcing for the ENSO model. The coupled model is described in section 3 and its stability properties are examined in section 4. Section 5 describes our experimental results, and the model response is compared to observations to assess which scenarios produce the most realistic response. Finally, we discuss the results and their implications in section 6.

2. Estimating stochastic forcing from observations

Our working hypothesis for this study is that the stochastic forcing is the internal variability of the atmosphere that is not correlated with variations in SST. The method used to estimate this component of the forcing is described in detail by Zavala-Garay et al. (2003); therefore we present only a brief description here.

a. Data

Weekly SST from Reynolds and Smith (1995) and daily surface winds from the National Centers for Environmental Prediction–National Center for Atmospheric Research (NCEP–NCAR) reanalysis project for the period 1982–99 were used to characterize oceanic and atmospheric variability, respectively, in the region 30°N–30°S, 112.5°E–80°W. The Reynolds SST data were interpolated from the original 1° × 1° grid to a coarse 2.5° × 2.5° grid using a weighted averaging scheme to match the NCEP wind data. In addition, the weekly SSTs were interpolated to daily values using cubic splines. The first three harmonics associated with the annual cycle were removed from both SST and wind and then anomalies were computed relative to a daily climatology.

b. Algorithm

As a first step, the component of covariability in SST and wind was determined using the singular value decomposition (SVD) approach described by Bretherton et al. (1992). The vectors of SST and surface wind anomalies will be denoted \( s(t) \) and \( w(t) = (U, V) \), respectively, and each vector represents the daily gridpoint values. For SST, \( s(t) \in \mathbb{R}^M \), where \( M = 68 \times 25 \) and \( \mathbb{R}^M \) is the set of real valued \( M \)-dimensional vectors, and for \( w(t) \in \mathbb{R}^{2M} \) for the two components of wind.

The sample covariance, \( C \in \mathbb{R}^{M \times 2M} \), between \( s \) and \( w \) is given by

\[
    C = (1/N - 1) \sum_{t=1}^{N} s(t)w(t)^T,
\]

where \( N \) is the total number of days. Only the first nine SVDs are considered here since they describe 96% of the total covariability. The singular vectors of \( C \), denoted \( \phi_j \), represent the coherent patterns of covariability between \( s \) and \( w \); thus, any information not captured by the singular vectors is not part of the covariance between the ocean and atmosphere and must be due to internal variability of the system components. Here we separate the wind into a component that covaries with SST and a component that describes the internal variability of the atmosphere.

Following Bretherton et al. (1992), the wind at any time can be expressed as a linear combination of the SVD basis functions \( \phi_j \) plus a remainder term, \( r(t) \):

\[
    w(t) = \sum_{j=1}^{9} a_j(t)\phi_j + r(t),
\]

where the coefficients \( a_j(t) \) are determined by projecting the wind data onto the corresponding SVD basis function. The first term on the rhs of (1) will be referred to as the coupled component of the wind anomaly. The remainder term \( r(t) \) represents the component of the wind that is not linearly correlated with SST and is the
sought-after stochastic component of the atmospheric forcing. This is a robust and relatively straightforward method for estimating the stochastic forcing, although there are other alternative and complementary approaches. For example, there is method of Linear Inverse Modeling (Penland and Magorian 1993), which can be used to estimate the noise characteristics in multivariate systems via the “fluctuation-dissipation” relation (Penland and Matrasova 1994). This approach, as well as those taken by DelSole and Hou (1999) and Sura (2003), are based on the work of Hasselmann (1988) on Principal Oscillation Patterns (POPs).

The stochastic forcing so derived will be used to force a coupled model. However, in order to obtain statistically significant model results, a time series much longer than 18 yr is required. Synthetic time series of stochastic forcing were therefore generated with the same statistics as \( r(t) \) in (1). As a first step, empirical orthogonal functions (EOFs) \( \Phi_i \) and principal component time series \( p_i(t) \) were constructed from \( r(t) \), where the first 25 EOFs account for 92% of its total variance. As suggested by the Akaike information criterion, a set of first-order autoregressive [AR(1)] models was used to extend the residual wind components in time. For each of the 25 EOFs, an AR(1) process \( a_i(t) \) was constructed using the lag-1 correlation \( \rho_i \) for the time series \( p_i(t) \) such that

\[
a_i(t + 1) = \rho_i a_i(t) + \xi(t),
\]

where \( \xi(t) \) is a Gaussian random variable with mean zero and unit variance. The lag-1 amplitudes \( \rho_i \) were approximately 1 week, and red noise time series of 500 yr were generated.

The resulting coupled part of the wind has maximum wind anomalies on the order of 5 m s\(^{-1}\) centered on or located just east of the date line. These coupled wind anomalies usually last several months. The stochastic forcing has highly variable winds, typically at about a magnitude of 6 m s\(^{-1}\), but with bursts as high as 15 m s\(^{-1}\). The stochastic forcing is concentrated in the western Pacific, west of the date line, and has a correlation time between 5 and 9 days. In the interest of brevity, see Zavala-Garay et al. (2003) for complete details of the method employed and a more detailed description of the resulting components of the forcing.

3. The coupled model

The model used in the present study is the intermediate coupled model of Kleeman (1993), which simulates the low-frequency component of the coupled ocean–atmosphere system of the Tropical Pacific. This model has been used for several years for ENSO forecasting and exhibits a high level of predictive skill (Kleeman et al. 1995). The model has also been used in numerous previous studies of ENSO (Kleeman 1993; Kleeman and Moore 1997; Kleeman and Power 1994; Kleeman et al. 1995; Moore and Kleeman 1996, 1997a,b, 1999a,b; Zavala-Garay et al. 2003). This is an anomaly model comprised of ½-layer ocean and atmosphere components on an equatorial \( \beta \)-plane. The atmospheric component is a Gill-type model that is slaved to the oceanic component. Atmospheric heating due to anomalous deep convection is parameterized using observed empirical relationships (Kleeman 1989) and is controlled by a moist static energy threshold corresponding to about 28.5°C, which is an important source of nonlinearity in the model. The thermocline depth is required to remain deeper than 22.5 m to prevent a runaway instability, another important source of nonlinearity in the model associated with the mixed layer dynamics. Model SST anomalies depend only on variations in the thermocline depth as described by Kleeman et al. (1992) and Kleeman (1993).

For the model configuration used here, where the ocean response is dominated by thermocline displacements, the dominant EOF of the ocean model SST is qualitatively similar to the dominant EOF of observed SST. When coupled to the atmospheric model, the hindcast skill is similar to that of other intermediate models (Kleeman 1993).

Many studies have demonstrated the power and utility of intermediate coupled models, like that used here, for elucidating ENSO dynamics and for seasonal prediction of ENSO indices such as SOI and Niño-3. While such models clearly do not describe the full range of atmospheric and oceanic processes at work, they do capture the dominant large-scale air–sea interaction processes that are thought to control ENSO, and which are believed to be primarily linear. The predictive skill of the Kleeman model and others of the same genre [e.g., the model of Zebiak and Cane (1987)] is a testament to this. In this respect, many intermediate coupled models still outperform complex fully coupled ocean–atmosphere general circulation models. Hence the intermediate model used here captures the important ENSO dynamics as evidenced by its predictive skill, but at a low computational overhead, allowing us to perform many experiments.

4. Stability of the coupled model

The stability properties of the coupled model depend primarily on the coupling strength between the ocean and atmosphere. In the model a linear drag law relates surface wind \( \mathbf{U} = (U, V) \) to wind stress, namely, \( \tau = \alpha \rho_c c_D \mathbf{U} \), where \( \alpha \) represents the coupling strength \( (\text{m s}^{-1}) \), \( \rho_c \) is the air density \( (1.2 \text{ kg m}^{-3}) \), and \( c_D \) is an
empirical drag coefficient \((1.5 \times 10^{-3})\). Numerical experiments reveal a Hopf bifurcation near \(\alpha = 7.2\). Figure 1 shows three 250-yr time series of the Niño-3 index from three model integrations, one performed below the bifurcation, one coincident with, and one beyond the bifurcation. After initialization with an initial perturbation, the model is allowed to run free, revealing its natural long-term behavior for each choice of \(\alpha\). Some transient behavior can be seen in the first 15–20 yr in each plot.

For \(\alpha \geq 7.2\) (Figs. 1b and 1c), the model changes from having a stable fixed point (Niño-3 index decays to zero) to having a limit cycle with a period of approximately 4 yr. The limit cycle at \(\alpha = 7.3\) is a regular oscillation with a 4-yr period and has a maximum positive Niño-3 SST anomaly (SSTA) bounded above by 2.5°C and a minimum of about −2°C (Fig. 1c). In addition there is a low-frequency modulation of the oscillation with a period of about 35 yr. The e-folding decay time of the Niño-3 index in Fig. 1a, for \(\alpha = 6.8\), is around 4 yr.

Figure 2 shows the probability distribution function (PDF) for the time series in Figs. 1b,c, constructed using a nonparametric technique by applying a Gaussian kernel estimator to the data with a theoretically optimal fixed bandwidth (Bowman and Azzalini 1997). The PDF in Fig. 2a for \(\alpha = 7.2\) has two peaks and is reminiscent of the PDF of a regular small-amplitude oscillation (e.g., a sinusoidal wave), although for the latter the peaks would have equal amplitude. For \(\alpha = 7.3\), Fig.

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**Fig. 1.** Niño-3 index (250 yr) from three coupled model experiments (a) in the stable regime, (b) at the Hopf bifurcation, and (c) in the unstable regime.
2b shows a wide and flat PDF because the model spends more time away from zero and attains larger values of the Niño-3 index more frequently.

5. Stochastic forcing experiments

Most previous studies of ENSO as a stochastically forced oscillation have focused primarily on additive stochastic forcing (i.e., forcing that is independent of ENSO amplitude and phase; Eckert and Latif 1997; Kleeman and Moore 1997; Moore and Kleeman 1999a; Penland 1996; Penland and Matrasova 1994; Thompson and Battisti 2000a,b; Zavala-Garay et al. 2003). In this section, we explore and compare the impact of additive and multiplicative stochastic forcing on ENSO variability in the coupled model.

a. Designing the multiplicative noise

Designing the appropriate multiplicative noise forcing for the coupled model is nontrivial because attempts to characterize intraseasonal variability, the dominant source of stochastic forcing, as a function of interannual variability using observations have been largely unsuccessful. Simply correlating indices of stochastic forcing with ENSO indices yields no statistically significant relationship. More sophisticated global MJO indices proposed by Slingo et al. (1999) and Hendon et al. (1999) do not capture the nature of the relationship either. While this is a rather controversial topic and an area of intense research, Kessler (2001) noted that it is not the global aspects of the MJO that are related to ENSO, but rather the meandering over the Pacific Ocean of the winds associated with the MJO that is related to ENSO’s interannual variability. Even though robust statistical relationships between MJO activity and ENSO have not been established, observational analyses hint strongly at relationships between the two (e.g., Keen 1987; Lukas 1987; McPhaden et al. 1998; Zhang 2001).

Since observations provide no clear empirical rela-
tionships to link MJO and ENSO, we resort to physical arguments instead. In this work several functional forms describing the amplitude of the multiplicative stochastic forcing in terms of the low-frequency component were considered. Using the Niño-3 (\(N_3\)) index as a measure of ENSO phase and amplitude, Perez (2003) tested several different polynomial forms of stochastic forcing amplitude based on \(N_3\) and found that the model response was relatively insensitive to the functional form. Therefore, in the present study we consider only the case in which the stochastic forcing amplitude is a linear function of \(N_3\).

The stochastically forced coupled model can be expressed symbolically as
\[
d\mathbf{m}/dt = \mathbf{X}(\mathbf{m}) + g(\mathbf{m}, t),
\]
where \(\mathbf{m}\) is the model state vector, \(\mathbf{X}\) is a nonlinear operator representing the model dynamics, and \(g(\mathbf{m}, t) = \text{str}(\mathbf{m}) \sum_{i=1}^{25} a_i(t) \Phi_i\) is the stochastic forcing. The red noise time series \(a_i(t)\) are defined in (2), and \(\text{str}(\mathbf{m})\) represents the stochastic forcing amplitude, which is a function of the model environment. A Stratonovich discretization was used to solve the stochastic differential equation since it represents a physically continuous process (Ewald et al. 2004), and the standard rules of Riemann integration apply (Gardiner 1985). The time step used in the model integration was 1 day, and the externally imposed noise has a decorrelation time of about a week.

Observations indicate that when \(N_3\) is large and positive (negative), then the easterly trade winds weaken (strengthen) because of surface warming (cooling) and a diminished (enhanced) zonal SST gradient. For example, when \(N_3 > 0\), the trades slacken, the warm pool spreads east, and WWBs are given an opportunity to be active farther east than during normal conditions (Keen 1987; Kessler et al. 1995; Lukas 1987). In this situation the impact of stochastic forcing would be more keenly felt by the ocean. The lower bound on \(\text{str}\) for \(\gamma N_3 < -1\) simulates the suppression of MJO-related stochastic forcing during cold events.

To simulate this behavior, stochastic forcing amplitude \(\text{str}\) was defined as a piecewise linear function of \(N_3\) according to
\[
\text{str} = \begin{cases} 
1 + \gamma N_3 & \text{for } \gamma N_3 \geq -1 \\
0 & \text{for } \gamma N_3 < -1,
\end{cases}
\]
where \(\gamma\) is taken to be a positive constant. For a non-zero value of \(\gamma\), \(g(\mathbf{m}, t)\) is clearly multiplicative. Additive forcing corresponds to the case \(\gamma = 0\) and \(\text{str} = 1\) for all \(N_3\).

b. Sensitivity to variations in \(\gamma\)

The sensitivity of the model response to \(\gamma\) in (3) was explored first by choosing \(\gamma = 0.25\degree, 0.5\degree, 1\degree, 2\degree,\) and \(3\degree\text{C}^{-1}\) and comparing these cases to the additive forcing case \((\gamma = 0\) and \(\text{str} = 1\) for all \(N_3)\). Figure 3 shows an example of the time evolution of \(\text{str}\) and \(N_3\) for the case \(\gamma = 1\). Our analysis (summarized below) revealed that
the choice $\gamma = 1$ does not represent a special case of multiplicative stochastic forcing, but is a reasonable choice.

The first four moments of 100-yr-long time series of the Niño-3 index were calculated for integrations of the coupled model run below, at and above the Hopf bifurcation (i.e., $\alpha = 6.8, 7.2$, and $7.3$, respectively). The mean was close to zero in each case. The largest variance occurred for the most unstable case examined, $\alpha = 7.3$, and variance decreased as $\gamma$ increased for every $\alpha$ but remained $\sim 10^{-14}C^2$. Skewness was similar in all three stability regimes and increased with $\gamma$. As the model became more unstable, the overall skewness increased slightly for each $\gamma$. Kurtosis was generally more positive and increased with $\gamma$ in all cases, which for the stable case means that the resulting PDFs had more substantial tails than for the unstable cases, indicating more frequent extreme values of the Niño-3 index in the stable cases. For each stability regime, the additive case had large kurtosis whereas small $\gamma$ values ($0 < \gamma < 1$) led to the smallest, in some cases even negative, kurtosis. The largest kurtosis was found when $\gamma = 3.0$ for all $\alpha$ considered.

We take the similarities between the first three moments of each time series as evidence that $\gamma = 1$ is a moderate choice for the dependence of the amplitude of the multiplicative forcing on the model Niño-3 index, and so our choice of $\gamma$ does not slant our relationship toward any presupposed outcome. Thus $\gamma = 1$ in all of the experiments reported below.

c. Additive forcing versus multiplicative forcing in different stability regimes

To establish the differences in model response to additive versus multiplicative stochastic forcing in the different stability regimes, the coupled model was run for 100 yr with str given by (3) for the multiplicative case ($\gamma = 1$) and str = 1 for the additive case ($\gamma = 0$).

Figure 4 shows three time series of the monthly Niño-3 index for cases where the model was run with additive stochastic forcing in the three different stability regimes near the bifurcation point with the same realization of additive stochastic forcing constructed from the observations. As $\alpha$ increased past the bifurcation point, both variance and skewness of the 100-yr time series increase. Figure 5 shows time series for the coupled model run with multiplicative forcing. In this case the variance increases with $\alpha$, but the skewness does not change significantly after the bifurcation.

Figure 6 compares the PDFs of Figs. 4 and 5, which were calculated by applying the Gaussian kernel estimator to the data with a theoretically optimal fixed bandwidth. All of the multiplicative cases have PDFs that are taller and narrower than the corresponding additive cases. In the stable regime (Fig. 6a), the model response to both kinds of stochastic forcing is similar in the tails and different near the center. The multiplicative case is nearly unimodal, while the additive case has a broader PDF with a slight dip near the top. Near the bifurcation point, Fig. 6b shows a change in the model response to the two types of stochastic forcing. In the additive experiment, the PDF reveals a shoulder or separation into two neighboring peaks with one flatter than the other. The skewness of the additive PDF in Fig. 6b (0.21) is nearly double that of the one in Fig. 6a (0.11). The response to multiplicative stochastic forcing also shows a loss of unimodality. In the unstable case, the PDFs (Fig. 6c) change only in that they exaggerate the features of Fig. 6b. Skewness remains small for the additive case in Fig. 6c (0.21), and the corresponding multiplicative case yields a PDF with skewness of 0.42, about twice that of the additive case result.

Figure 6 illustrates that multiplicative forcing near the bifurcation point and of unstable regimes alters the ENSO variability of the model in that the multiplicative forcing acts to constrain the model response to smaller-amplitude negative anomalies. The additive stochastic forcing encourages greater variability and excites more extreme events. This is evident if we notice that the PDFs constructed from the model response to multiplicative forcing of the near-bifurcation (Fig. 6b) and unstable (Fig. 6c) cases have wider tails indicating more excursions toward extreme Niño-3 values. This is most noticeable in Fig. 6 for La Niña-type conditions, as the additive forcing produces a cooler-than-average Niño-3 index more frequently than multiplicative forcing and more frequently as the stability parameter is increased.

This behavior is a consequence of the form of the amplitude function (3). In the additive case warm and cold events are subject to the continual influence of stochastic forcing; hence, both types of events may be amplified compared to the case when there is no stochastic forcing (e.g., Kessler and Kleeman 2000). In the case of multiplicative forcing (3), when $\gamma N_\delta < 1$, the stochastic forcing amplitude is zero, meaning that moderate and strong cold events are not subject to the constant influence of the stochastic forcing. In contrast, warm events are always subject to the stochastic forcing with an amplitude that increases with $N_\delta$. Nonlinearity prevents a runaway instability, and a point is reached where the warm event becomes insensitive to the stochastic forcing (Moore and Kleeman 1999a). Thus, in the multiplicative case stochastic forcing always acts to enhance warm events, but not cold events, thereby skewing the PDF.
d. Forced versus unforced behavior

Comparing Fig. 6 to the PDFs of unforced integrations in Fig. 2, we can see that both types of forcing dramatically change the PDFs of the underlying system. At the bifurcation point (Figs. 2a and 6b), the noise forcing acts to smooth and broaden the bimodal PDF. In a physical sense, the bimodal PDF (Fig. 2a) shows that when the model is close to the bifurcation point, a small positive (0.05°C) and a small negative (−0.1°C) Niño-3 value are produced more frequently than any other. Beyond the bifurcation point (Figs. 2b and 6c) the PDF changes drastically from a wide, low multimodal PDF of the model’s inherent limit cycle to a PDF with a higher peak centered near zero. The tall peak centered about zero in Fig. 6c represents a system that stays near normal SST most of the time but occasionally yields larger-magnitude Niño-3 values, both positive and negative. The PDFs appear to be weakly bimodal in both cases for the unstable regime (Fig. 6c).

In the case of additive stochastic forcing, all cases have the same fixed points or limit cycles as the unforced cases in Fig. 2. For the stable case the stochastic forcing constantly pushes the system away from the fixed point as it tries to return to the resting state. Close to the bifurcation point and in the unstable case, the

![Fig. 4. 100-yr time series of the monthly Niño-3 index of the coupled model subject to additive forcing (a) in the stable regime, (b) at the bifurcation point, and (c) in the unstable regime, in the limit cycle.](image)
stochastic forcing constantly disrupts the regular oscillation of the limit cycle changing the most likely state of the system. In contrast, in the case of multiplicative forcing, all cases have different fixed points and limit cycles compared to those of the unforced case, and may in fact be continually changing. This implies a fundamental change in the oscillation characteristics of the system as reflected by the subtle changes in shape of the PDF.

e. Comparison with observations and robustness of results to length of record

It is of obvious interest to compare the model Niño-3 PDF with that of the observations. However, the reliable observed instrumental record is only 50 yr long, and the reliability of information about the PDF computed from such a short record containing so few large ENSO episodes is questionable. To quantify such uncertainties relative to the model experiments, we divided each 500-yr integration into 10 individual time series of 50-yr duration and compared all such realizations with the observations. For reference, Fig. 7 shows the PDF of the observed record and its deviation from a Gaussian distribution. The observations are unimodal, positively skewed (0.93), and clearly non-Gaussian. Figures 8a and 8b show the resulting PDFs of the model Niño-3 index for each 50-yr segment for the additive and multiplicative forcing cases, respectively, with \( \alpha = 7.2 \). Also shown is the PDF of the observed Niño-3 for comparison. All of the PDFs in Fig. 8a appear weakly bimodal, while those in Fig. 8b are a mix-

![Figure 5](image-url)
ture of bimodal and unimodal. In both cases all of the PDFs are positively skewed, but less so than the observations.

Figures 9a and 9b compare the autocorrelation of the model and observed Niño-3 index for the $\alpha = 7.2$ case of additive and multiplicative forcings, respectively, and reveal temporal information that the PDFs obscure. The autocorrelations in Fig. 9 all exhibit a similar initial drop in correlation with increasing lag up to month 10 similar to the observations, although beyond this there is considerable variability.

Figures 10a and 10b catalog the statistics of the 10 realizations in the form of boxplots for the $\alpha = 7.2$ case of additive and multiplicative forcings, respectively. The horizontal line in the middle of each box is the median. The top and bottom margins of a box represent the interquartile range, and the stems on the box indicate the maximum and minimum values of the data. Box 11 corresponds to the entire 500-yr model run, and box 12 shows the statistics of the observed Niño-3.

The interquartile range is the difference between the upper and lower quartiles. The upper (lower) quartile is the value halfway between the maximum (minimum) and the median. It is a robust measure of spread about the median in a dataset and thus another measure of the variance (Wilks 1995).
Fig. 7. Heavy line indicates the PDF of observed monthly Niño-3 (1950–2001). The normal reference band (enclosed by the light solid lines) indicates the way in which the PDF of observed data deviates from a Gaussian distribution (dashed line).

Fig. 8. Observed PDF (dashed) and 10 model PDFs (solid) of 50-yr runs subject to (a) additive and (b) multiplicative stochastic forcing ($\alpha = 7.2$).
both cases, all of the variability of the first 10 datasets, as indicated by the height of the box and the length of the stems, is captured by the entire 500-yr run as expected. None of the time series capture all of the observed variability; thus, the most extreme natural events are not reproduced by the model in either the additive or multiplicative case. The median of the observations is more negative than that of any of the model simulations, although most of the simulations have negative medians.

Figure 11 shows the sample means of the 10 time series along with their standard error for the $\alpha = 7.2$ additive (Figs. 11a,b) and multiplicative (Figs. 11c,d) forcing cases. For time series longer than 30 months, the standard error in the mean is given by $\sigma_{\text{mean}} = (\sigma / \sqrt{N})$, where $\sigma$ is the standard deviation and $N$ is the length of the sample time series (Young 1962). In both cases the sample means in Figs. 11a,c hover about the sample mean of the entire 500-yr time series, and in most cases the model is closer to the observed mean than the 500-yr mean. In both cases the standard error (Figs. 11b,d) for all realizations is much greater than the error in the entire 500-yr record because the standard error decreases with sample size $N$ as $(1/\sqrt{N})$. The standard error is a maximum for the observations because the variance of the observations is larger than any of the model integrations (see Fig. 10). It is an indication of the good forecast skill of the model (Kleeman et al. 1995) that, in general, each model realization and the observations have a similar standard error, within $10^{-2}$ °C.

In Fig. 12 we compare PDFs of the observed 50-yr record with those of the 500-yr model runs using additive and multiplicative stochastic forcing. Neither case brings our model closer to the truth. The model PDF tails are too flat because the model cannot reproduce the extreme events in the real system. In both cases the model produced unimodal and nearly normal PDFs. As a check on the visual comparison, we applied the Kolmogorov–Smirnov test, which is a strict and robust nonparametric test. The results indicate that the PDFs of both model runs are definitely not from the same distribution as the observations. They are also different from each other, although the D-statistic is an order $10^{-1}$ smaller than the D-statistic found when comparing either model run to the observations. (A smaller D-statistic means that the distributions are more similar.)

To establish confidence in the PDF comparisons between PDFs, we took the 500 yr of model Niño-3 values and resampled the data (bootstrap with replacement) to construct 50 time series that were each 50 yr long. We then formed 50 PDFs using the same nonparametric method as before. Figures 13 and 14 summarize the result of this procedure by showing the rms difference between the single PDF from the observations (Fig. 7) and the two ensembles of bootstrapped PDFs from the
stochastic forcing experiments. Figure 13 indicates that the two model datasets are more similar to each other than they are to the observations.

We calculated the rms difference over restricted portions of the domain in Figs. 13b,c with the intention of characterizing the model’s capacity to recreate near-normal or climatological behavior (peaks) and extreme events (tails). Figure 13b shows that the model responded to additive and multiplicative forcing in much the same way. The stars and circles frequently overlap, and occasionally the additive case is more unlike the observations than the multiplicative case. Contrast this with Fig. 13c, where the anomalously warm and cold response in the additive case is consistently closer to the observations than in the multiplicative case.

Figure 14 provides a useful reanalysis of Fig. 13c by

![Boxplots of observations (box 12) and model (boxes 1-11) subjected to (a) additive and (b) multiplicative stochastic forcing (α = 7.2). The first 10 boxplots (solid) correspond to ten 50-yr runs of the model. Boxplot 11 (dash-dot) represents the distribution of the entire 500-yr model run. Box 12 (dashed) is the distribution of the observed monthly Niño-3.]
separating the left (La Niña) and right (El Niño) tails of the PDFs for comparison. The model response to additive forcing consistently, but only slightly, outperforms the multiplicatively forced cases. Here the rms difference is actually very small, on the order of $10^{-2}$ °C, and is similar between all three measurements. In Fig. 14b, the multiplicative stochastic forcing as implemented in this experiment [Eq. (3)] does a poor job simulating the La Niña events. We see that this confirms the result of a qualitative look at Fig. 12; revisiting the PDFs here we see that the left tail of the additive model run for 500 yr falls halfway between the observations and the multiplicative run.

The rms difference in Fig. 14b is larger than in Fig. 14a, and points directly to the issue of choosing a multiplicative forcing function that takes all of the relevant physics into account correctly. As noted in section 6c, a negative value of $N_3$ less than $-1$ turns off the stochastic forcing ($\text{str} = 0$) so the development of moderate and strong cold events are not aided by the stochastic forcing. It is probable that this particular form of the stochastic forcing amplitude discounts too much the interactions between the La Niña state of the ocean and higher-frequency atmospheric processes. Although it is a reasonable way to express the relationship between WWBs and the background state of SST, this multipli-
Cative formulation cannot account for nuances of the tropical atmosphere–ocean relationship that are not fully understood at this time.

The analysis of this section shows that our comparison of 50 yr of observations with multiple 50-yr model integrations produces consistent results. Therefore, we are confident that it is possible to form reliable conclusions about the differences in model response to both additive and multiplicative stochastic forcing in spite of the visually striking variability of the PDFs in Figs. 8a,b.

6. Discussion and conclusions

The aperiodicity of ENSO motivated our investigation into the effects of additive versus multiplicative stochastic forcing on an intermediate coupled model of ENSO. In this work we have attempted to give a new perspective on how the natural tropical Pacific atmosphere and ocean might operate by expanding on the seminal work of Hasselmann (1976), who proposed additive stochastic forcing as a model and null hypothesis for climate variability. We emphasize that the findings presented here are the result of forcing a particular ENSO model with stochastic forcing derived from observations using one specific method.

Convincing empirical evidence in support of an identifiable relationship between the stochastic forcing and ENSO variability remains elusive. Instead we used physical arguments to construct a multiplicative stochastic forcing in which the amplitude of the stochastic forcing is a linear function of the Niño-3 index and is intended to mimic the observation that warm (cold) ENSO events have a tendency to enhance (suppress) MJO-related activity, recalling the feedback loop proposed by Zhang (2001). The effect on the model of changing from additive to multiplicative stochastic forcing was found to be more significant than the difference in model response to a variety of multiplicative forcings both linear and nonlinear.

It has been argued that the coupled ocean–atmosphere system in the tropical Pacific operates near its bifurcation point (Fedorov et al. 2003); therefore, we ran the coupled model near its bifurcation. In the unstable regime, beyond the model's Hopf bifurcation, the Niño-3 PDFs were fundamentally altered when the forcing was changed from additive to multiplicative. This is in agreement with the theory of stochastic systems (Horsthemke and Lefever 1984) and the related findings of Perez (2003) that multiplicative stochastic forcing can modify the PDF of a dynamical system by continually changing the behavior and characteristics of the oscillatory solutions. To our knowledge, this is the first time this effect has been documented in a complex geophysical model like that considered here.
The Niño-3 response of the model to multiplicative forcing and additive forcing was compared with observations in segments of 50-yr intervals to match the length of the observational record. There was great variation in the shape of the PDFs of the different 50-yr model responses to both types of forcing. Many ENSO models do not reproduce the extreme events that are seen in nature, and even when run for 500 yr, the coupled model cannot reproduce the observed variability. In fact, the longer the model is integrated with either type of forcing, the more normally distributed the response becomes, essentially obeying a fuzzy form of the central limit theorem. However, for each type of forcing, the similarity between the the 50-yr and the 500-yr integrations show that for the model, length of record is not crucial when testing for the first two moments of the underlying PDF. The sample means and standard errors of the 50-yr model runs compare favorably to those of the observations, assuring us that the coupled model can reasonably approximate the mean and variance of 50 yr of the observed Niño-3 index.

The model response to multiplicative forcing results in better reproduction of the observed mean and median over all and is more skewed to positive Niño-3
index (warm events), like the observations, than the additive case, for all of the 50-yr realizations. We found that additive forcing produces a slightly larger variance in the model response (on the order of 0.1 at the most) than the multiplicative forcing, which means that it is closer to the observed variance. The 500-yr time series of model Niño-3 subject to additive forcing reproduces the negative anomalies or La Niña statistics better than the multiplicative cases. This is because the additive stochastic forcing can reinforce the development of both warm and cold anomalies alike while the multiplicative forcing used here acts to enhance only the development of warm anomalies, but not moderate and strong cold anomalies. The result is that multiplicative forcing produces larger positive anomalies, and additive forcing produces larger negative anomalies. Neither form of forcing can reproduce both of the most extreme natural events, either El Niño or La Niña.

The variability in shape among the 10 PDFs of 50-yr model integrations suggests that it is difficult to state with certainty from the observed PDF whether nature is additively or multiplicatively forced. Therefore, we quantified the variation between model PDFs by bootstrapping (resampling with replacement) 50-yr datasets from the 500-yr model time series. This made it possible to test a large number of model PDFs against the 50 yr of Niño-3 observations. Both types of forcings reproduce the peak of the PDF of observations with similar accuracy; however, the more interesting behavior is in the tails, which quantify the frequency of extreme warm and cold events. We found that while the additive case was more like the observations when the PDF was

![Fig. 14. Comparison of the same ensemble of model PDFs to observations and each other as in Fig. 13, but with a more restricted domain. Here the rms difference is averaged over the (a) right and (b) left tails of the PDFs separately.](image-url)
taken as a whole, a careful examination showed that this was almost completely due to its better performance in the cold events. In the real world, the effect of stochastic atmospheric events on the ocean does not disappear when the eastern tropical Pacific is anomalously cold (when the Niño-3 index is very negative). Thus, one way to improve the simulation of cold events using multiplicative noise might be to make it more realistic by changing the form of the amplitude of the stochastic forcing, $st$, making it small but nonzero in order to reinforce those cold anomalies.

In our experiments, we used only one method to construct our basic noise forcing and one coupled ocean–atmosphere model. To understand exactly how this ENSO study is limited by these factors will require further investigation into the sensitivity of results to the procedures applied here. It is possible that by using a more complex coupled ocean–atmosphere model along with a more sophisticated form of multiplicative stochastic forcing, the model PDF could be altered even more dramatically when the additive noise forcing is changed to a multiplicative form.

In this work, we applied stochastic forcing to an intermediate coupled model using a rudimentary, but arguably realistic, feedback mechanism for the interactions between the tropical Pacific ocean and atmosphere on a range of subdecadal time scales. This feedback was implemented via a multiplicative stochastic forcing in which the atmosphere not only forces the ocean, but also responds to it. We showed that by using multiplicative stochastic forcing instead of additive, one can quantitatively change the dynamics of the system. We hope to have provided a positive impulse in the new direction of using multiplicative stochastic forcing in the modeling of ENSO as an alternative to the standard, and less physical, additive approach.

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