Atmospheric Dynamics

Lecture 8: Linearization Part 3 (Baroclinic Instability)

1 Background

In the previous two Lectures we discussed linearization about a state of rest. However it is clear from the discussion of the thermal wind relation and the observed jet streams that such a linearization is not appropriate for the mid-latitudes. In particular we saw that geostrophy plus the radiative forcing demands both vertical and horizontal shears in the mean state of the atmosphere. Such features are well known from fluid dynamics to induce turbulence and the atmosphere is no different in that respect. It turns out (for reasons that are not yet completely understood) that linearization is a very useful tool in understanding the turbulence observed. This turbulence is essentially what is commonly thought of as weather.

2 Two level model

Many of the important features of the linearization we seek can be obtained by considering the atmosphere as a two vertical level system by which we mean that vertical derivatives are replaced by appropriate finite difference approximations. Consider the quasi-geostrophic equations in pressure coordinates (Lecture 5) and define the stream function as

$$\psi \equiv \frac{1}{f_0}\Phi$$

then the vorticity equation derived in Lecture 5 can be rewritten as

$$\frac{\partial \nabla^2 \psi}{\partial t} + \boldsymbol{u}_g \cdot \nabla(\nabla^2 \psi) + \beta \psi_x = f_0 \omega_p \tag{1}$$

where the geostrophic relations are

$$u_g = -\psi_y$$
$$v_g = +\psi_x$$

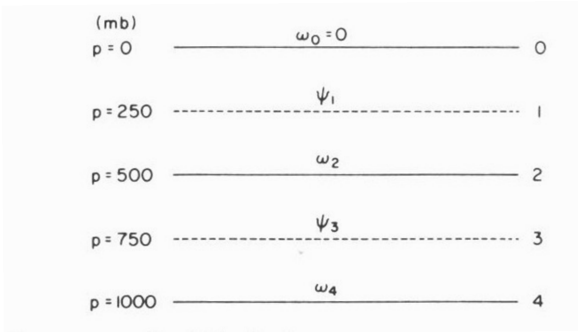

Figure 1: Vertical levels for two level model

and this flow is along the contours of the stream function. The temperature equation when combined with the hydrostatic relation gives

$$\frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial p}\right) + \boldsymbol{u}_g \cdot \nabla \left(\frac{\partial \psi}{\partial p}\right) = -\frac{\sigma}{f_0}\omega \tag{2}$$

where $\sigma = \frac{R}{p}\overline{S}(p)$ i.e. is related to the mean static stability of the atmosphere. To set up the two level model we evaluate the stream function and vertical velocity at the levels shown in Figure 1. . As an approximation we set $\omega_4 = 0$ [1]. Using the obvious discretization of vertical derivatives in equations (1) and (2) we obtain the three equations

$$\begin{array}{c} \frac{\partial \nabla^2 \psi_1}{\partial t} + \boldsymbol{u}_1 \cdot \nabla(\nabla^2 \psi_1) + \beta \frac{\partial \psi_1}{\partial x} = \frac{f_0}{\delta p}\omega_2 \\ \frac{\partial \nabla^2 \psi_3}{\partial t} + \boldsymbol{u}_3 \cdot \nabla(\nabla^2 \psi_3) + \beta \frac{\partial \psi_3}{\partial x} = -\frac{f_0}{\delta p}\omega_2 \\ \frac{\partial}{\partial t}(\psi_1 - \psi_3) + \frac{1}{2}(\boldsymbol{u_1} + \boldsymbol{u_3}) \cdot \nabla(\psi_1 - \psi_3) = \frac{\sigma \delta p}{f_0}\omega_2 \end{array} \tag{3}$$

where $\delta p = 500mb$. This constitutes a closed set of equations known as a two level model.

3 Baroclinic instability problem

We study now the implications of a vertically sheared zonal velocity mean state i.e. the basic situation pertaining to the jet stream. We assume therefore a mean state that is constant horizontally (and purely zonal in velocity)

[1]This has the effect in the linearized equations of setting barotropic shallow water speed to an infinite value which is a reasonable assumption when studying slower effects.

but not vertically. The appropriate mean stream function for this is

$$\begin{aligned}\overline{\psi}_1 &= -U_1 y \\ \overline{\psi}_3 &= -U_3 y\end{aligned}$$

Linearizing equations (3) about this mean state and assuming for simplicity solutions only dependent on x and t we obtain

$$\begin{aligned}\left(\frac{\partial}{\partial t} + U_1\frac{\partial}{\partial x}\right)\frac{\partial^2\psi_1}{\partial x^2} + \beta\frac{\partial\psi_1}{\partial x} &= \frac{f_0}{\delta p}\omega_2 \\ \left(\frac{\partial}{\partial t} + U_3\frac{\partial}{\partial x}\right)\frac{\partial^2\psi_3}{\partial x^2} + \beta\frac{\partial\psi_3}{\partial x} &= -\frac{f_0}{\delta p}\omega_2 \\ \left(\frac{\partial}{\partial t} + U_m\frac{\partial}{\partial x}\right)(\psi_1 - \psi_3) - U_T\frac{\partial}{\partial x}(\psi_1 + \psi_3) &= \frac{\sigma\delta p}{f_0}\omega_2\end{aligned}$$

where $U_m \equiv \frac{1}{2}(U_1 + U_3)$ is the vertically averaged mean flow and $U_T = \frac{1}{2}(U_1 - U_3)$ is the component of the mean flow due to the thermal wind relation and so is called the thermal wind. We can define analogous quantities for the stream function ψ_m and ψ_T (which we call respectively the barotropic and baroclinic stream function). If we add the first two equations we obtain a new equation. Additionally if we subtract them and use the third equation to remove ω_2 then we get the two equations involving only the stream function

$$\begin{aligned}\left(\frac{\partial}{\partial t} + U_m\frac{\partial}{\partial x}\right)\frac{\partial^2\psi_m}{\partial x^2} + \beta\frac{\partial\psi_m}{\partial x} + U_T\frac{\partial}{\partial x}\left(\frac{\partial^2\psi_T}{\partial x^2}\right) &= 0 \\ \left(\frac{\partial}{\partial t} + U_m\frac{\partial}{\partial x}\right)\left(\frac{\partial^2\psi_T}{\partial x^2} - 2\lambda^2\psi_T\right) + \beta\frac{\partial\psi_T}{\partial x} + U_T\frac{\partial}{\partial x}\left(\frac{\partial^2\psi_m}{\partial x^2} + 2\lambda^2\psi_m\right) &= 0\end{aligned}$$

where $\lambda^2 \equiv f_0^2/\left(\sigma(\delta p)^2\right)$. Note the resemblance of these equations to those derived in the previous Lecture for the Rossby waves. Note however how the non-zero mean state modifies the form non-trivially. As for that case we seek Fourier solution

$$\psi_m = A\exp(i(kx - \omega t) \qquad \psi_T = B\exp(i(kx - \omega t) \tag{4}$$

and obtain the following two dispersion equations

$$\begin{aligned}ik\left[\left(\frac{\omega}{k} - U_m\right)k^2 + \beta\right]A - ik^3U_TB &= 0 \\ ik\left[\left(\frac{\omega}{k} - U_m\right)(k^2 + 2\lambda^2) + \beta\right]B - ikU_T(k^2 - 2\lambda^2)A &= 0\end{aligned} \tag{5}$$

These equations only have solutions for the Fourier amplitudes A and B if the determinant of the linear equation here vanishes i.e. if

$$(\omega/k - U_m)^2 k^2 \left(k^2 + 2\lambda^2\right) + 2(\omega/k - U_m)\beta\left(k^2 + \lambda^2\right) + \left[\beta^2 + U_T^2k^2\left(2\lambda^2 - k^2\right)\right] = 0$$

This quadratic equation can be solved for $\omega/k = c$ the phase velocity with the two solutions

$$c = \frac{\omega}{k} = U_m - \frac{\beta\left(k^2+\lambda^2\right)}{k^2\left(k^2+2\lambda^2\right)} \pm \sqrt{\delta}$$

with the potentially negative (and interesting) δ given by

$$\delta = \frac{\beta^2\lambda^4}{k^4\left(k^2+2\lambda^2\right)^2} - \frac{U_T^2\left(-k^2+2\lambda^2\right)}{\left(k^2+2\lambda^2\right)}$$

Note that the possibility that δ is negative depends critically on the mean thermal wind U_T. If δ is indeed negative it follows rather obviously that exponentially growing solutions in time can occur. If the thermal wind vanishes then the two solutions for ω are

$$\omega = U_m k - \beta k^{-1}$$
$$\omega = U_m k - \tfrac{\beta k}{k^2+2\lambda^2}$$

These are the dispersion relations for Rossby waves advected by the mean flow U_m. The second term on the right hand sides is identical to the form derived in the previous Lecture[2] with particular choices for shallow water speeds: the first is infinite (because of the lower boundary condition), the second close to that of the first baroclinic mode. The vertical structure of the first can be shown to be barotropic and the second baroclinic.

In the physically interesting case that $U_T \neq 0$ it is useful to consider when the critical parameter $\delta = 0$ i.e.

$$\frac{\beta^2\lambda^4}{k^4\left(k^2+2\lambda^2\right)} = U_T^2\left(-k^2+2\lambda^2\right)$$

This gives a curve in the (U_T, k) plane that separates stable and unstable modes and is called the neutral curve. We can solve this equation easily for k and plot as a function of the mean thermal wind U_T and this is shown in Figure 2.

Note that as U_T increases instability also does and that there is a minimum value for the unstable region that selects a particular value of k i.e. it selects a particular scale. It turns out that this value of k is also when

[2]with $l = 0$ since we are dealing with solutions constant in the y direction

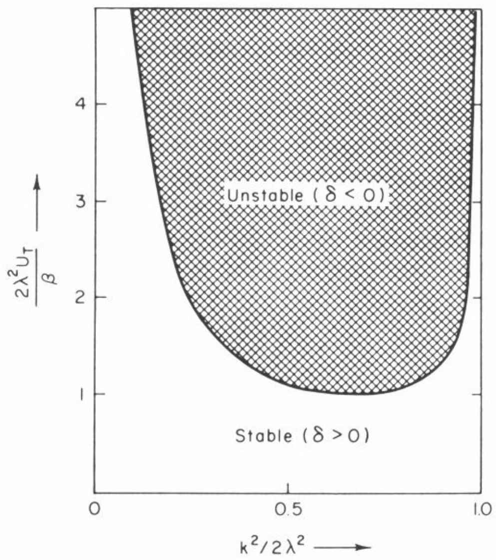

Figure 2: The neutral curve for the two layer quasi-geostrophic model linearized about a vertically varying mean state.

instability is greatest for a given U_T. Elementary calculus shows this occurs when

$$k = \sqrt[4]{2}\lambda$$

and this corresponds to a particular value of c the phase velocity. The level at which this matches the mean velocity is termed by synopticians the "steering" level since the mean flow at that level appears to be moving the disturbance. With realistic values for λ the horizontal scale corresponding to this most unstable wave is about $4000km$ which corresponds well with the observed scale of weather systems. It is natural to focus on the most unstable modes since they will tend to dominate the total solution given sufficient time. In general notice how the instability region tends to select out a natural space scale and through the dispersion relations a natural time scale. These scales correspond well with the observed phenomena. The growth rate of the unstable waves may be shown to a strong monotonic function of the mean thermal wind. For realistic choices for U_T many unstable linear modes exist i.e. we are high on the plot in Figure 2. In fact the minimum unstable mode occurs when the shear is about $8ms^{-1}$ which is well below the typical observed jet stream shear. The time scale associated with the growth of the

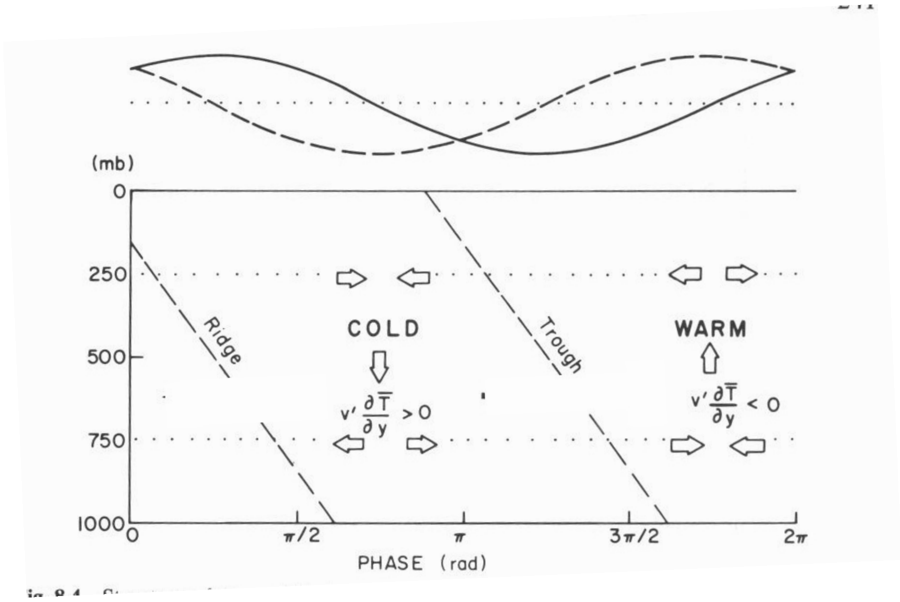

Figure 3: Structure of the most unstable mode. The disturbance moves from W to E. The dashed curve is 500mb temperature perturbation which precedes in time the geopotential (solid curve).

unstable modes can of course be derived from ω and for realistic model values is about $1 - 2$ days which again corresponds with storm development in the atmosphere.

The solutions for k and ω can be resubstituted into equations (5) and then (4) to reconstruct the particular horizontal solution at the various vertical levels. The corresponding vertical velocities, temperature (and geopotential) are easily obtained from the linearized quasi-geostrophic equations. The solutions for the most unstable modes show a very characteristic structure which is sketched qualitatively in Figure 3.

This wave moves from left to right i.e. from west to east. This is due to the mean state being so directed. Positive temperature perturbations occur first and are associated with upward motion (i.e. usually rain) and negative geopotential (i.e. low pressure at a given height). Positive geopotential perturbations (high pressure) occur later and are associated with downward motion (no cloud) and cold temperatures. The tilt with height of geopotential is also a notable (and realistic) feature of the unstable mode.

The reader will recognize the above pattern as that often occurring in a mid-latitude storm. Of course the analysis above is limited since it is linear and the non-linear “saturation” of the above unstable waves has been well studied. In general a baroclinic life cycle has been noted in both models and observations (see [1]). Many questions still remain however concerning the full broad spectrum non-linear problem which constitutes a very particular turbulence configuration.

References

[1] A.J. Simmons and B.J. Hoskins. The Life Cycles of Some Nonlinear Baroclinic Waves. *J. Atmos. Sci.*, 35:414–432, 1978.