Atmospheric Dynamics

Lecture 7: Linearization Part 2

1 Linear shallow water equations

In the previous Lecture we discussed linearization about a state of rest and found that convenient solutions could be obtained using a seperation of variables between the vertical and the horizontal plus time. The latter fields give then the linear shallow water equations with the shallow water speeds being a function of the vertical Strurm Liouville system eigenvalues. These equations have three different kinds of solutions: Gravity/Poincare waves, Rossby waves and Kelvin waves. In the mid-latitudes only the first two are important while the latter is often important in the equatorial region. We discuss here the former two and defer a general discussion until a later Lecture.

To keep the discussion general we consider the compressible equations in pressure coordinates discussed in the first section of Lecture 5. We choose the following separation of variables:

$$\begin{aligned}
\overrightarrow{u} &= \widehat{u}(p)\overrightarrow{U}(x,y,t) \\
\Phi &= \widehat{\Phi}(p)\eta(x,y,t) \\
T &= \widehat{T}(p)\tilde{T}(x,y,t) \\
\omega &= \widehat{h}(p)\widetilde{w}(x,y,t)
\end{aligned} \tag{1}$$

Substitution into the full equations gives us the vertical equations

$$\begin{gathered}
\widehat{u} = \widehat{\Phi}/g \\
\widehat{\Phi}_p = -\tfrac{R}{p}\hat{T} \\
\hat{h}_p = \hat{u}/H_0 \\
\hat{T} = S(p)\hat{h}
\end{gathered}$$

and a final second order PDE of

$$\hat{h}_{pp} + \frac{RS(p)}{pc^2}\hat{h} = 0$$

with $c^2 = gH_0$.

The shallow water equations are identical to the incompressible case i.e.

$$\begin{aligned} U_t - fV &= -h_x \\ V_t + fU &= -h_y \\ h_t + c^2(U_x + V_y) &= 0 \end{aligned}$$

where $h = g\eta$.

The so called Poincare/gravity waves can be obtained by setting the Coriolis parameter $f = f_0$ a constant. Add the x derivative of the first equation and the y derivative of the second and take the t derivative of the resulting equation; then use the difference of the y derivative of the first and the x derivative of the second with the third equation and combine this with second t derivative of the third equation to obtain

$$h_{ttt} - c^2(h_{xx} + h_{yy})_t + f_0^2 h_t = 0$$

Consider now a general wave like solution of the form

$$h = h_0 \exp(i(\omega t - kx - ly) \tag{2}$$

we obtain the dispersion relation

$$\omega(\omega^2 - c^2(k^2 + l^2) + f_0^2) = 0$$

The solution with zero frequency corresponds to a stationary geostrophic flow. The other solutions give the Poincare waves

$$\omega^2 = c^2(k^2 + l^2) + f_0^2$$

These are hyperboloids in the ω, k, l space and a minimum frequency of f_0 occurs when the solutions have zero wavenumbers in the x and y directions i.e. they are constant. This corresponds with a period of a half a day. All other gravity waves have a shorter period and a group velocity of c the so-called shallow water speed.

2 Rossby waves

There are other lower frequency wave solutions of the shallow water equations that are a consequence of the variation of the Coriolis parameter. An easy way to obtain these solutions is via the quasi-gesotrophic equations from

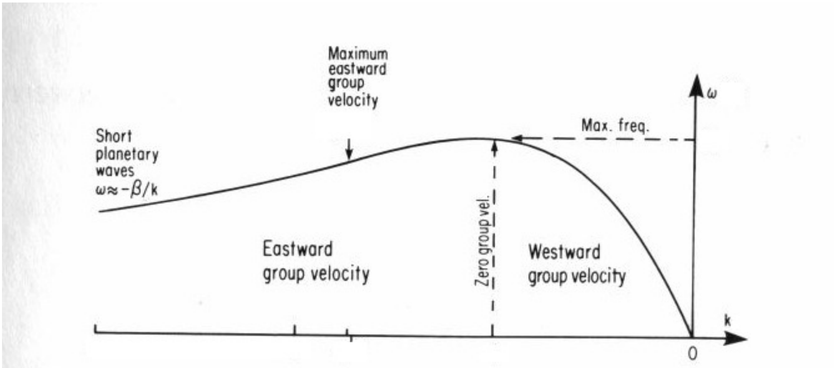

Figure 1: Dispersion diagram for Rossby waves. The plot is for zonal wavenumber frequency space.

Lecture 5. The prognostic momentum equations (6) can be combined with the geostrophic relations (5) to give the two equations

$$f_0^2 v_a = -\Phi_{yt} - \beta y \Phi_x$$
$$f_0^2 u_a = -\Phi_{xt} + \beta y \Phi_y$$

Taking the divergence of these two equations gives

$$\frac{\partial \omega}{\partial p} = \frac{1}{f_0^2}\left[\Phi_{xxt} + \Phi_{yyt} + \beta \Phi_x\right]$$

If we now perform the vertical separation of variables from the previous section we obtain for the horizontal part of the geopotential

$$\frac{1}{c^2}\eta_t = \frac{1}{f_0^2}\left[\eta_{xxt} + \eta_{yyt} + \beta \eta_x\right]$$

Substituting the Fourier ansatz (2) gives the Rossby wave dispersion relation

$$\omega = \frac{-\beta k}{k^2 + l^2 + \frac{f_0^2}{c^2}}$$

Notice the dependence of this relation on the shallow water speed which indicates that there are different waves for the different vertical eignemodes. A plot of this Rossby wave dispersion relation in the ω, k plane is shown in Figure 1.

Notice that for Rossby waves of small wavenumber (long wavelength) the group propagation is westward while the large wavenumber (short wavelength) waves propagate to the east. The stationary point occurs at

$$k^2 = l^2 + \frac{f_0^2}{c^2}$$

The small wavenumber approximate dispersion relation is

$$\omega = -\frac{\beta c^2}{f_0^2} k$$

while the approximate dispersion relation for large wavenumbers is simply

$$\omega = -\frac{\beta}{k}$$

The peak frequency for typical values of c and f_0 is considerably smaller than the minimum frequency for gravity waves showing typically a rather large spectral gap. Rossby waves are thus excited by low frequency forcing in the atmosphere (order days or longer) while gravity waves are forced by high frequency events (period order hours). If we define the distance parameter

$$a \equiv \frac{1}{\sqrt{l^2 + \frac{f_0^2}{c^2}}}$$

then the maximum frequency is easily shown to be $\omega_{max} = \beta a/2$ when $k = 1/a$. Notice that this distance parameter which is sometimes called the Rossby radius decreases with latitude and increases with the shallow water speed c.

The Rossby waves are present in the full shallow water equations along with the gravity waves (more detail in a later lecture). The quasi-geostrophic approximation therefore has the effect of filtering out the gravity waves and retaining the slowly varying distrubances. This was of great value in the early days of weather prediction when gravity waves were a large problem in initial conditions since these were less than ideal. In the presence of non-zero mean flows such as the jetstream the analysis above is of course modified. The first order effect is simply to shift the waves group velocity by the background flow. For low frequency waves this can mean that their group velocity which is westward for no background flow can become eastward. Many observations in both atmosphere and ocean have yielded strong evidence for Rossby waves.

3 Thermal wind relation

The spherical nature of the Earth implies that the amount of solar radiation impinging on the surface is essentially proportional to the cosine of latitude and hence decreases sharply from equator to pole. This basic radiative effect implies strong meridional gradients of temperature at least close to the surface. The gradient is carried into the interior of the atmosphere by a variety of advection and mixing processes. The geostrophic balance then implies that there must be a consequent wind flow. If we combine the hydrostatic equation with the geostrophic balance condition (Lecture 5 first section) for zonal velocity we obtain in pressure coordinates

$$(u_g)_p = \frac{RT_y}{p}$$

which is known as the thermal wind relation since it implies that a meridional temperature gradient is balanced by a vertical zonal wind gradient. Since the wind vanishes at the surface this implies a zonal flow aloft. This is known as the jetstream. Figure 2 shows the northern winter zonally averaged temperature and wind. Notice that the meridional temperature gradient is greatest near the surface consistent with its radiative origin. Note also that it reverses in the stratosphere and this means that the thermal wind peaks at the top of the troposphere. It also peaks in the mid-latitudes since this is where the equator to pole temperature gradient naturally peaks.

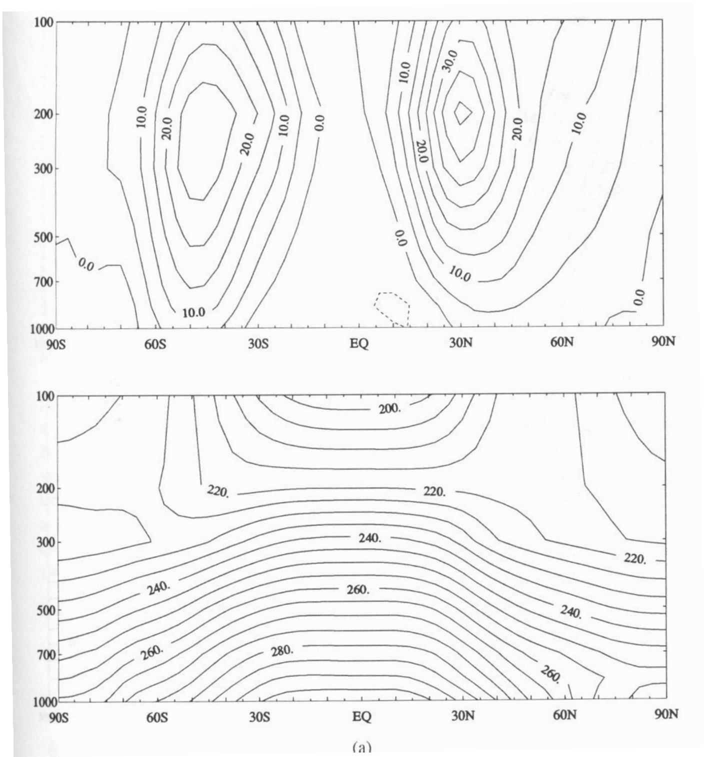

Figure 2: The northern winter zonally averaged zonal wind (top panel) and temperature (lower panel). Notice the accuracy of the thermal wind relation. Units are meters per second and Kelvin respectively.