Climate Dynamics

Lecture 6: Equatorial ocean adjustment

# 1 Shallow water equation interpretation

As we have seen in previous lectures the shallow water equations govern the horizontal flow for small perturbations from a state of rest. In general in the tropics most kinds of forcing project more strongly onto the first few baroclinic modes than onto the barotropic mode and therefore adjustment of the fluids to equilibrium takes place mainly through the agency of these relatively slow moving modes. Modes beyond the first 3-4 tend to be relatively highly damped because of their complex vertical structure. This means that such waves generated by a particular forcing tend not to propagate far before the large damping causes them to dissipate. Thus a large amount of the dynamic behavior of the equatorial ocean is carried by the first 3-4 vertical modes and study of the shallow water equations for them is highly revealing for how full $3D$ adjustment to equilibrium takes place. In terms of physical quantities of interest it is obvious that $(U, V)$ are contributors to horizontal currents. Less obvious is the interpretation of $h$. Careful rereading of the vertical mode decomposition carried out previously however reveals that $h$ corresponds to vertical displacements within the fluid. For the baroclinic modes most of this vertical motion takes place in the interior of the fluid and is most noticeable at the interface between surface waters and the deep ocean (the thermocline). Thus the most common identification of $h$ is with thermocline displacements.

# 2 The Sverdrup balance

It is useful to consider steady state solutions of the shallow water equations for wind forcing:

$$\begin{aligned} -\beta y V &= -h_x + X \\ \beta y U &= -h_y + Y \\ U_x + V_y &= 0 \end{aligned} \tag{1}$$

Differentiating the first equation by $y$ and the second by $x$ and use of the third equation gives the relation

$$\beta V = Y_x - X_y \tag{2}$$

which is commonly referred to as the *Sverdrup balance*. It states that the poleward currents are proportional to the curl of the wind stress. This relation works surprisingly well in explaining the interior surface flow of much of the world's oceans. A further interesting result is obtained if we consider the case of a constant zonal wind stress (a situation that applies approximately in the equatorial waveguide of the Pacific). Equation (2) shows that $V = 0$. The third equation of (1) then shows that $U$ cannot vary zonally which implies it must be zero in a basin situation. The first equation gives the useful relation

$$h_x = X$$

which shows that the slope of the thermocline is proportional to the zonal wind stress. Such a result again is borne out to a surprising degree of accuracy over much of the equatorial ocean. In the Pacific it implies that the thermocline slopes up from west to east and that most of this slope occurs in the central part of the basin where zonal wind stress is greatest. The solutions $(U, V) = 0$ appear less promising and it will turn out that the inviscid balance of equation (1) is not appropriate for currents where dissipation effects are quite important.

# 3 Initial spin up: The Yoshida jet

The previous section considered equilibrium behavior. Next we consider what happens when a wind stress patch is suddenly turned on. This kind of situation is surprisingly common in equatorial situations: Two important applications are the sudden appearance of winds associated with the Indian monsoon in the equatorial Indian ocean and the sudden onset of a "westerly wind burst" in the equatorial western Pacific.

Consider the situation for which a uniform patch of zonal wind stress $X$ is turned on suddenly. Until the effects of the boundaries are felt (see below) the solution will be zonally uniform and so we can set variations with $x$ to zero in the shallow water equations. The non-dimensional equations thus read:

$$\begin{aligned} U_t - yV &= X \\ V_t + yU &= -h_y \\ h_t + V_y &= 0 \end{aligned} \tag{3}$$

Take the time derivative of the second equation and combine it with the

$y$ derivative of the third equation:

$$V_{tt} + yU_t = -h_{yt} = V_{yy}$$

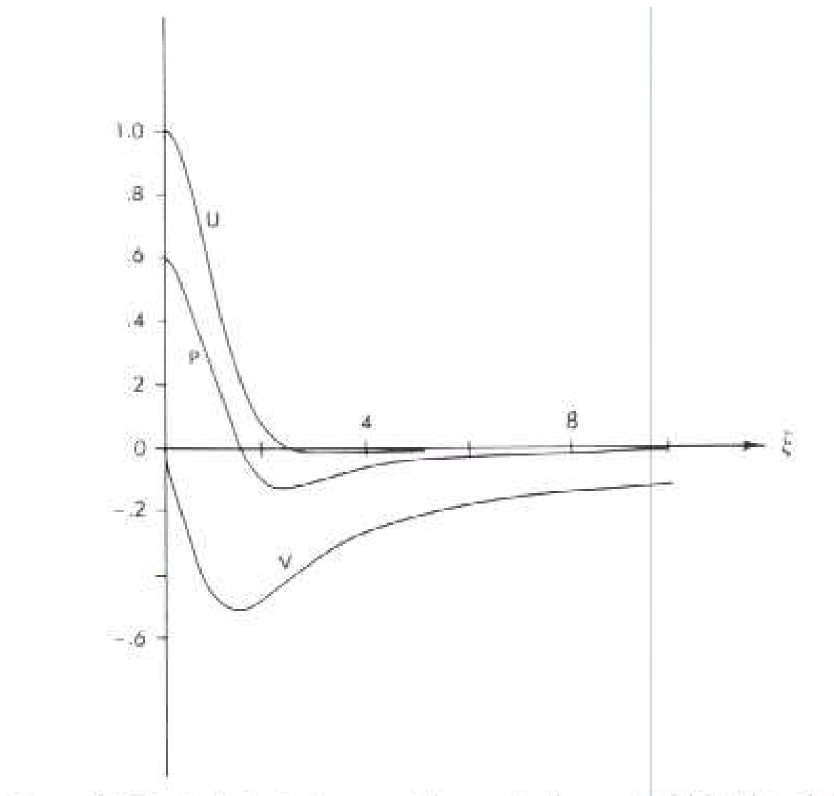
Finally use the first equation to eliminate the term involving $U$ :

$$V_{tt} + (\partial_{yy} - y^2)V = yX$$

Now the time scale with which we non-dimensionalized the equations is of order a couple of days for the leading baroclinic modes. Thus for time scales much longer than this (e.g. 20 days) we can neglect the first term in this equation and obtain

$$(\partial_{yy} - y^2)V = yX \tag{4}$$

The solution of this ODE $V_p$ is known and is the so-called Lommel function of order $\frac{1}{4}$ which is sketched in Figure 1.



Substituting this solution back into the first equation of (3) shows that $U$ increases uniformly with time and is given by

$$U = (yV_p - X)t$$

and similarly substitution into the third equation gives a uniformly increasing solution for $h$:

$$h = -V_{py}t$$

the time independent factors for these two variables are also displayed in Figure 1. Thus the solution to a uniform patch of zonal wind stress suddenly switched on is a constantly accelerating zonal jet which is confined to within a radius of deformation or so of the equator. Such jets have often been observed in the equatorial ocean in response to the sudden appearance of significant zonal wind stress in the atmosphere. When a model of the shallow water equations is spun up from rest the Yoshida jet is the first observed response.

# 4 Transient Solution

## 4.1 Long wave approximation

Recall that the non-dimensional shallow water equations may be written as

$$\begin{aligned} U_t - yV &= -h_x + rX \\ V_t + yU &= -h_y + rY \\ h_t + (U_x + V_y) &= -sQ \end{aligned}$$

Now if the time derivative in the second equation is neglected then the Sturm Liouville basic equation reduces to

$$V_{yy} - \left(\frac{k}{\omega} + y^2\right) V = 0$$

and therefore the dispersion relation becomes simply

$$-\frac{k}{\omega} = (2n + 1)$$

which is the equation for long wave Rossby modes. The Kelvin wave is still obviously a solution since it has $V = 0$ in any case. Thus by this device we have filtered out gravity and short wave Rossby modes which are not important to climate time scale adjustment. For historical reasons this approximation is called "long-wave" even though it is also a low frequency approximation as well. In terms of the ancillary variables $q$ and $r$ as well as the parabolic cylinder function decomposition, the forced equations become

$$\begin{aligned} q_t^n + q_x^n - V^{n-1} &= rX^n - sQ^n \\ r_t^n - r_x^n + 2(n+1)V^{n+1} &= -rX^n - sQ^n \\ 2(n+1)q^{n+1} - r^{n-1} &= 2rY^n \end{aligned} \tag{5}$$

where the term involving $V$ in the final equation has been dropped as a result of our approximation. Not surprisingly given the long wave equations non-dispersive nature, these equations are easily solved. Eliminating $r^n$ from the second equation using the third gives us

$$2(n+2)\left(q_t^{n+2} - q_x^{n+2}\right) + 2(n+1)V^{n+1} = -rX^n - sQ^n + (\partial_t - \partial_x)\, 2rY^{n+1}$$

or

$$2n\left(q_t^n - q_x^n\right) + 2(n-1)V^{n+1} = -rX^{n-2} - sQ^{n-2} + (\partial_t - \partial_x)\, 2rY^{n-1}$$

Multiplying the first equation through by $2(n-1)$ and adding then gives

$$(2n-1)q_t^n - q_x^n = r\left(-X^{n-2}/2 + (n-1)X^n\right) + s\left(-Q^{n-2}/2 - (n-1)Q^n\right) + (\partial_t - \partial_x)\, rY^{n-1} \tag{6}$$

where solutions are valid for $n = 0, 1, 2, \ldots$ providing that terms with negative superscripts are dropped. Free solutions of these equations are simply arbitrary disturbances travelling with a velocity of $-1/(2n-1)$. They are thus easily identified with Kelvin and Rossby wave parts of the general solutions to the long wave shallow water equations. Note that when $n = 1$ we get the long wave vestige of the mixed Rossby-gravity mode which is just a forced solution ($q^1 = rY^0$). Note that each "Rossby solution" also has other "parts" because of the third equation of (5) and the non-vanishing of $V^n$. In other words there is a non-zero $r^{n-2}$ simply related to $q^n$ as well as a $V^{n-1}$ which are required to reconstruct the solution for each Rossby mode.

A very useful simple case (which we consider below) is provided by the case when the only forcing term is $X^0$. This corresponds to a Gaussian zonal wind stress patch with width of the equatorial radius of deformation. This example illustrates many of the basic principles of equatorial adjustment.

For this case there are only two forced equations for $q^n$ i.e. with $n = 0, 2$. The Kelvin wave equation is the first and is simply

$$q_t^0 + q_x^0 = rX^0 \tag{7}$$

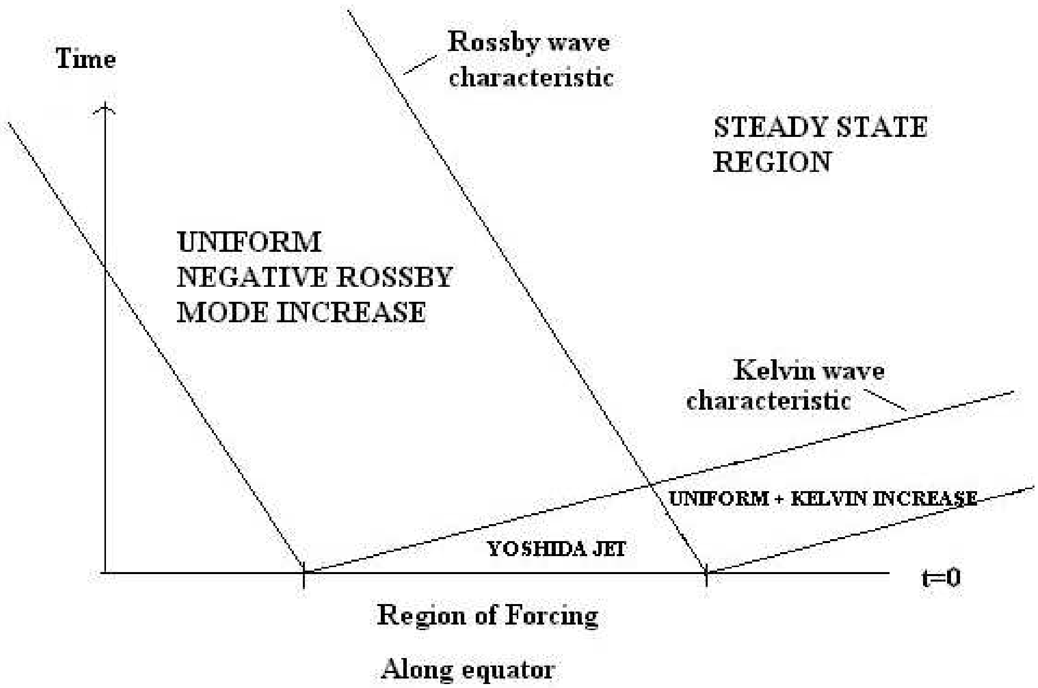
There is no corresponding $r^{-2}$ here so we just get the complete Kelvin solution by substituting for the definition of the full meridional $q$ and the meridional structure for $u$ and $h$ is clearly just proportional to $D_0(y)$. For $n = 2$ we obtain

$$\begin{array}{rcl} 3q_t^2 - q_x^2 & = & -rX^0/2 \\ r^0 & = & 4q^2 \\ V^1 & = & \frac{4}{3}q_x^2 - \frac{1}{6}rX^0 \end{array} \tag{8}$$

All other components for $q, r$ and $V$ are unforced. Reconstruction of the first Rossby mode using $D_0(y)$ and $D_2(y)$ results in the solution displayed in Lecture 4 if we assume a simple structure zonally for $X$. Note also that because the thermocline perturbation $h$ is given for this Rossby mode by $q^2(y^2 + 2)\exp(-y^2/2)$ it follows that a given zonal windstress spins up opposite signed Rossby and Kelvin waves. This fact is fundamental to equatorial ocean dynamics as we shall see in a future lecture.

## 4.2 Unbounded domain spinup

As a first cut at the spinup problem we consider the case of steady finite zonal forcing with meridional structure given by $D_0(y)$ and we shall assume that the spinup takes place in an infinite domain with the forcing switched on at time zero. The solution of this problem is a straightforward application of the forced equations (7) and (8). The configuration is illustrated schematically in Figure 2 which is an example of a Hovmoller diagram ($x - t$ diagram).




Characteristics are drawn for Kelvin and Rossby waves emanating from either end of the forcing. Between these lines it is easily shown that amplitudes for $q^0$ and $q^2$ spinup uniformly at a rate given by the forcing terms in the respective equations. Once outside the lines there is no longer any acceleration and the amplitudes reach a steady state (they are connected by characteristics to zero forcing regions). The value of the amplitudes therefore in the steady region can be computed by uniformly increasing $q^0$ and decreasing $q^2$ for the appropriate amount of time given by the Figure. Thus the amplitude for $q^0$ within the forcing region is easily seen to be

$$q^0_{steady} = rX^0 x$$

while the amplitude for $q^2$ which decreases more slowly but for a longer period is

$$q^2_{steady} = -rX^0(L-x)/2$$

The steady values outside the forcing region are constant with $x$ and are obviously just the same as the edge points by continuity. Substituting these into the equations for $h$ and $U$ gives

$$\begin{aligned} h_{steady} = \quad (q+r)/2 \quad &= \quad \left((q^0+r^0)D_0 + q^2 D_2\right)/2 \quad = rX^0\left((3x-2L)D_0/2 - (L-x)D_2/4\right) \\ U_{steady} = \quad (q-r)/2 \quad &= \quad \left((q^0-r^0)D_0 + q^2 D_2\right)/2 \quad = rX^0\left((2L-x)D_0/2 - (L-x)D_2/4\right) \end{aligned}$$

Along the equator this becomes

$$\begin{aligned} h_{steady}(0) &= x - L/2 \\ U_{steady}(0) &= 3L/2 - x \end{aligned}$$

Note that the thermocline perturbation is in Sverdrup balance with negative perturbations to the west and positive ones to the east. The solution for $U$ is not the Sverdrup one as the forcing is finite in extent. There is a steady equatorial jet in the direction of the forcing. If the forcing is made uniform meridionally it may be shown that the steady zonal jet becomes zonally uniform.

## 4.3 Boundary effects

The spinup becomes more complex in the (usual) presence of western and eastern boundaries. Consider first the eastern boundary where (by definition) the zonal flow must vanish. This means that the ancillary variables satisfy

$$q = r$$

Expanding in terms of parabolic cylinder functions we obtain a similar equality for the components and if we assume that the meridional forcing is not large then we obtain from the third equation of (5) the following recursive relation:

$$q^n = 2(n+2)q^{n+2} \tag{9}$$

Now at the eastern boundary, as we saw in Lecture 4, we have the reflection of a Kelvin wave as Rossby modes. This means that $q^0$ is known and the amplitudes for the Rossby modes may be calculated by iteration from (5) and are given by

$$q^n = \frac{1}{2^n (n/2)!} q^0$$

where $n$ is even as the reflected modes are, of course, symmetric about the equator.

At the western boundary things are more difficult if one makes the long wave approximation since short wave Rossby modes have been filtered and these are an essential component of the reflection of a Kelvin wave. Fortunately one is able to demonstrate that in the frequency range for which the long wave approximation holds that the short wave Rossby modes satisfy the condition

$$\int_{-\infty}^{\infty} U^{SW} dy = 0$$

which of course means that a similar relation must hold for the long waves as well. Expanding this condition in terms of cylinder functions once again gives the condition

$$\sum_{n=0}^{\infty} \left(q^n - 2(n+2)q^{n+2}\right) I_n = 0 \tag{10}$$

where

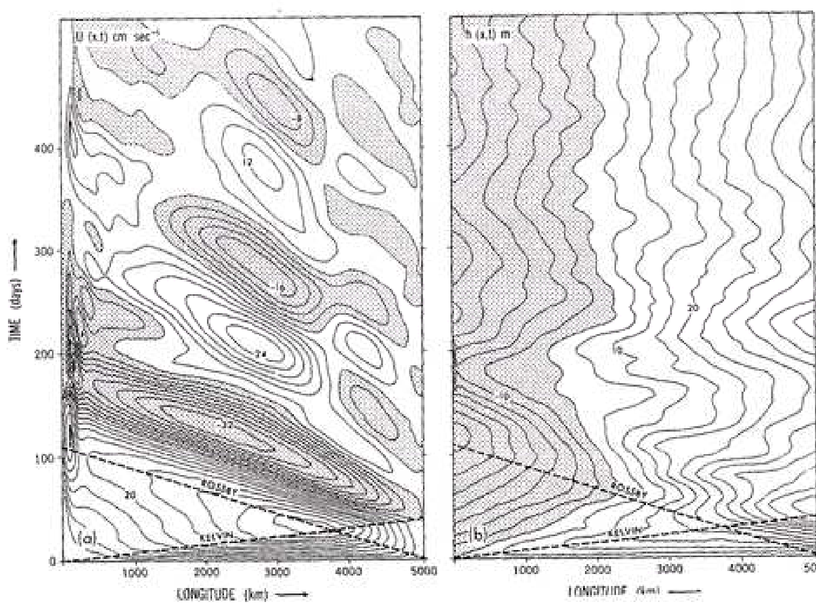
$$I_n = \int_{-\infty}^{\infty} D_n(y) dy$$

may be expressed in terms of factorials. We are interested in the amplitude of the reflected Kelvin wave which can easily be obtained from (10):

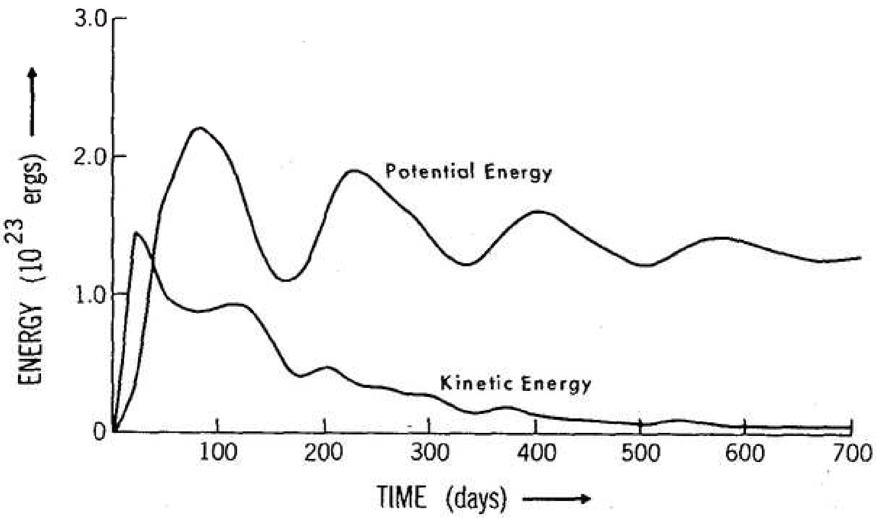
$$q^0 = -\sum_{n=1}^{\infty} q^{2n} \left(I_{2n} - 4nI_{2n-2}\right) \tag{11}$$

i.e. it is expressed in terms of the incident symmetric Rossby waves.

Now the complete spin up process could be in principle solved for analytically and this has been done in order to construct closed form simple models of El Nino. The expressions obtained are complex and here we confine ourselves to a more pictorial presentation. Illustrated in Figure 3 is a Hovmoller diagram of $h$ and $U$ for the adjustment to a uniform zonal wind stress patch which covers the entire basin.




Notice the repeated reflections of Kelvin and Rossby waves from both boundaries and their large effects. What is significant however is the gradual approach to the Sverdrup equilibrium state. This damped oscillatory behavior is a consequence of the leakage of energy at both boundaries. At the western boundary this is due to the dissipation of reflected short wave Rossby modes while in the east it is a consequence of Rossby waves propagating to higher latitudes as a so-called coastal Kelvin wave. This process is summarized in Figure 4 which shows the damped oscillation of the equatorial potential and kinetic energies.

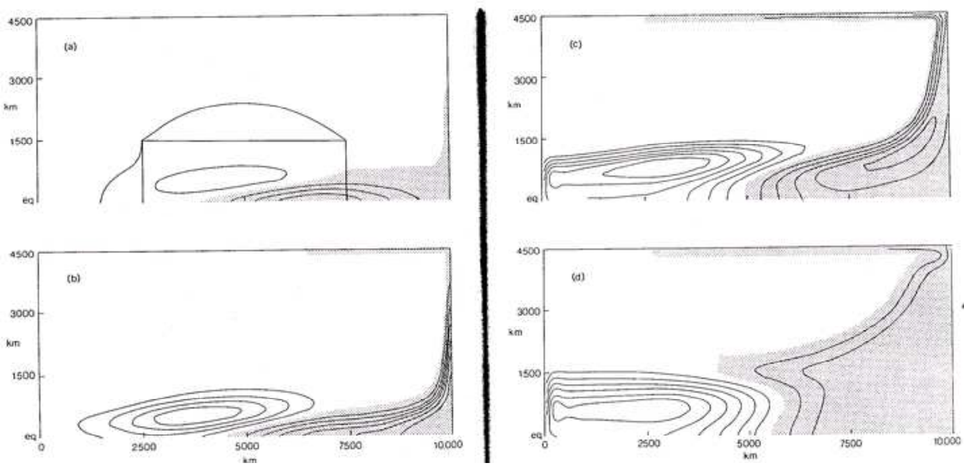



Some thought concerning this reflection process and recognition of the fact that at the equator most of the eastern boundary reflection is carried by the first symmetric Rossby wave leads us to the period of the oscillation in this Figure as

$$T = 4L/c \tag{12}$$

where $L$ is the basin width and $c$ is the Kelvin wave speed. This is the basin transit time for a Kelvin wave followed by a first mode Rossby wave. For the Pacific and for a first baroclinic mode we obtain $T \sim 250$ days. Notice though that the complete adjustment takes considerably longer than this i.e. on the order of $2-3$ years. These time scales we shall see are connected with the period of El Nino.

A horizontal plot of the adjustment process may be seen in Figure 5 which shows the response to a non- uniform zonal windstress patch at various times into the adjustment.




Notice the large propagation of Rossby mode energy into higher latitudes at the eastern boundary. Because these modes travel very slowly the adjustment time increases considerably at higher latitudes as an obvious generalization of the formula (12) to higher order Rossby modes shows. In the mid-latitudes this time can be of the order of a decade or more and in fact it was claimed a few years ago that the residual of the huge 1982-83 El Nino could be seen in the North Pacific in the mid 1990s.

The spinup process seen in Figure 5 may also be noted in the onset of El Nino events because here a large zonal wind stress anomaly typically develops centred on the equator at the international dateline. Displayed in Figure 6 is a plot of sea level anomalies during July 1997 as the 97/98 El Nino was spinning up.