Climate Dynamics

Lecture 4: Shallow water equations in the tropics

1 Solution on the equatorial β-plane

In the tropics it is a reasonable approximation to set the Coriolis parameter $f = 2\Omega \sin \varphi$ to the approximate expression

$$f = \beta y$$

where $\beta = 2.3 \times 10^{-11} m^{-1} s^{-1}$ is exact at the equator and approximately constant for most of the tropics. y is the distance from the equator. The shallow water equations now read

$$\begin{aligned} U_t - \beta y V &= -h_x \\ V_t + \beta y U &= -h_y \\ h_t + c^2 (U_x + V_y) &= 0 \end{aligned} \tag{1}$$

These may be reduced to a single equation in V by straightforward albeit somewhat complex manipulation (see Gill, Chapter 11). The result is

$$(V_{xx} + V_{yy})_t + \beta V_x - c^{-2} V_{ttt} - \frac{\beta^2 y^2}{c^2} V_t = 0 \tag{2}$$

General solutions to these equations can be found by separating variables again and can be written as sums of terms of the form

$$V = V(y) \exp(ikx - i\omega t) \tag{3}$$

where we need to derive a dispersion relation for k and ω. Substitution of (3) into (2) leads to the equation

$$V_{yy} + \left(\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega} - \frac{\beta^2 y^2}{c^2} \right) V = 0 \tag{4}$$

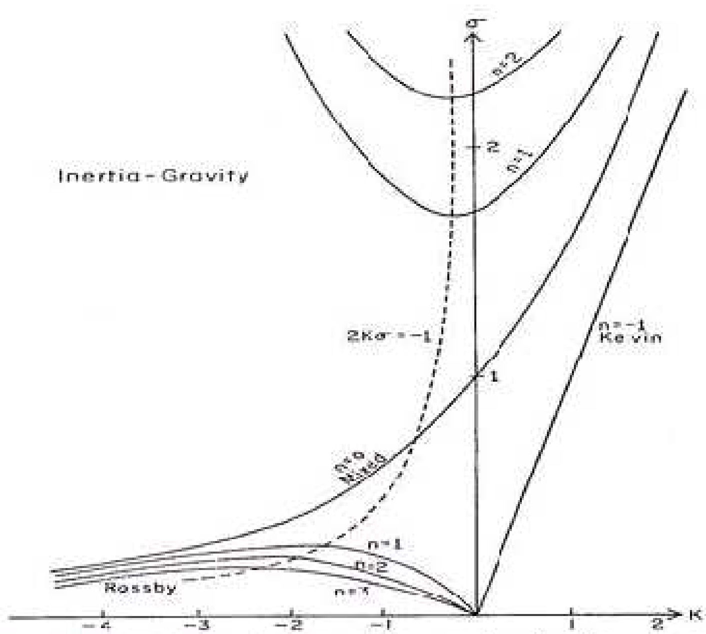
Now the operator $-\partial_{yy} + y^2$ is easily shown to be Hermitian and positive and we again have a Sturm Liouville problem this time on an infinite domain. The solutions to this eigenproblem are well known (those of you who have done quantum mechanics will recognize this equation as the Schroedinger wave equation for the harmonic oscillator which has been much studied). The eigenvalues are discrete and (obviously) bounded below and we have

$$\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega} = \frac{\beta}{c}(2n+1) \quad for \quad n = 0, 1, 2, 3...... \tag{5}$$

In addition there is a solution which has $V = 0$. Substitution of this in (1) gives the dispersion relation $\omega^2 = c^2 k^2$ and the negative root here gives unbounded solutions for U and h as $y \to \infty$ so we are left with the additional relation

$$\omega = kc \tag{6}$$

The solutions to both dispersion relations are plotted in Figure 1.



Dispersion diagram for equatorially trapped modes. The unit of frequency is $(\beta c)^{1/2}$ and the unit of zonal wave number k is the inverse of the radius of deformation $(c/\beta)^{1/2}$. [From Cane and Sarachik (1976).]

The upper (i.e. high frequency) curves are gravity waves (qf Lecture 3); the curves on the lower left hand side are called *Rossby waves* while the curve corresponding to (6) is for the so-called *Kelvin wave*[1]. We shall consider all these solutions in more detail below.

[1]The other solution which has $n = 0$ is called a *mixed Rossby-gravity wave* since it resembles the former for $k \to -\infty$ and the latter for $k \to \infty$.

The eigensolutions of (4) (as well as the additional $V = 0$) are the well known *parabolic cylinder functions*

$$D_n(\xi) \equiv D_n(\sqrt{\beta/c y}) = \exp\left(-\xi^2/2\right) H_n(\xi) \tag{7}$$

where $H_n(\xi)$ are the *Hermite polynomials* for which we have the following

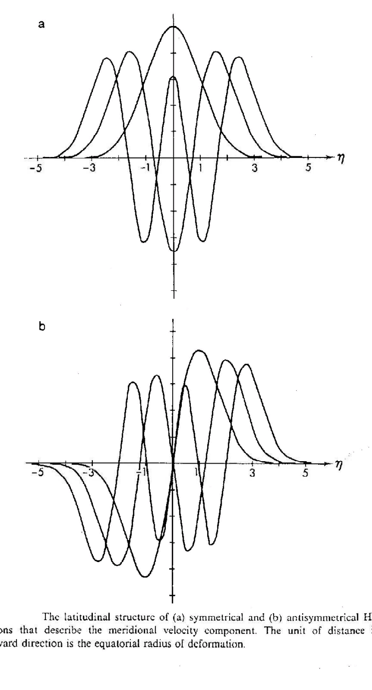
$$\begin{array}{rcl} H_0(\xi) & = & 1 \\ H_1(\xi) & = & 2\xi \\ H_2(\xi) & = & 4\xi^2 - 2 \\ \frac{\partial H_n}{\partial y} & = & 2nH_{n-1} \\ \xi H_n & = & nH_{n-1} + 0.5H_{n+1} \end{array} \tag{8}$$

The eigensolutions satisfy the orthogonality relation

$$\int_{-\infty}^{\infty} D_n(\xi) D_m(\xi) d\xi = 2^n n! \pi^{1/2} \delta_{mn}$$

Note that we have introduced a natural scaling distance $a = \sqrt{c/\beta}$ which is known as the equatorial radius of deformation.

The first few parabolic cylinder functions are displayed in Figure 2.



The latitudinal structure of (a) symmetrical and (b) antisymmetrical Hermite functions that describe the meridional velocity component. The unit of distance in the northward direction is the equatorial radius of deformation.

Note that they symmetric (about the equator) for n even and antisymmetric for n odd. Solutions for V are proportional to these functions while the solutions for U and h can be obtained by substituting for the V solution back in (1) and using (3), (8) and (7):

$$\begin{array}{rcl} V & = & C\exp\left(ikx - i\omega t\right) D_n(\xi) \\ U & = & C\exp\left(ikx - i\omega t\right)\left[\frac{n^{1/2}D_{n-1}(\xi)}{\omega+ck} + \frac{(n+1)^{1/2}D_{n+1}(\xi)}{\omega-ck}\right] \\ h & = & C\exp\left(ikx - i\omega t\right)\left[\frac{n^{1/2}D_{n-1}(\xi)}{\omega+ck} - \frac{(n+1)^{1/2}D_{n+1}(\xi)}{\omega-ck}\right] \end{array} \tag{9}$$

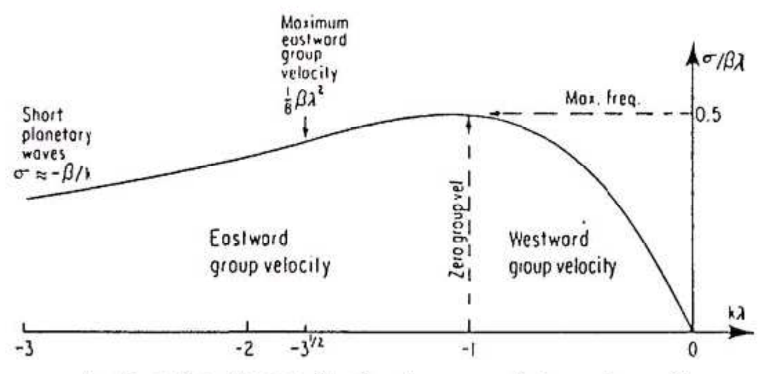
Notice that when U and h are symmetric then V is antisymmetric and conversely. The Kelvin wave solution has $V = 0$ and U and h proportional to $D_0(\xi)$ (see below for further detail). We now consider the solutions in detail.

2 Rossby waves

The low frequency solutions of the dispersion relation (5) are the Rossby waves and depend for their existence on the fact that the Coriolis parameter varies with latitude (i.e. that $\beta \neq 0$). Their approximate dispersion relation can be obtained by neglecting the term involving ω^2 in (5) and solving for ω :

$$\omega = \frac{-\beta k}{k^2 + \frac{\beta(2n+1)}{c}} \tag{10}$$

A schematic of this relation is depicted in Figure 3.

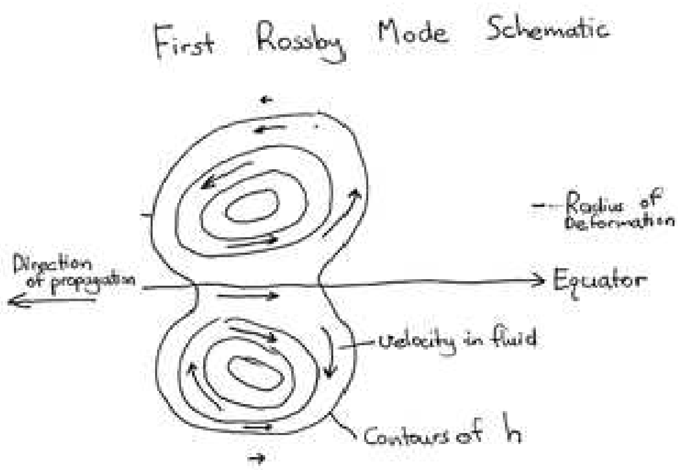


A dispersion diagram for Rossby waves that can be written as $\sigma/\beta\lambda = -k\lambda/(1 + (k\lambda)^2)$, where σ is frequency and $\lambda = c/f$ is the radius of deformation. [From Gill (1985).]

The limits of $|k|$ small (long waves) and $|k|$ large (short waves) are of significant interest physically. In the former case we can neglect k^2 from the denominator in (10) and obtain

$$\omega = \frac{-ck}{2n+1} \tag{11}$$

which is nondispersive and indicates that these waves propagate towards the west. Note that as n increases this speed slows. The first (or gravest) symmetric Rossby wave has $n = 1$ and the long wave horizontal structure is plotted in Figure 4.



Note the cyclonic couplet structure. It travels at a speed which is one third of the shallow water velocity. These waves play a fundamental role in equatorial dynamics and can easily be discerned in the observations of the equatorial ocean where they have typical zonal spatial scales of $2000 - 5000km$.

The approximate dispersion relation for the short waves is obtained by neglecting the other term in the denominator of (10):

$$\omega = -\beta/k \tag{12}$$

Differentiating this relation with respect to k shows that these (dispersive) waves have group velocity $c_g = \beta/k^2$ which shows that they propagate toward the east. As can be seen by examining the denominator in (10) these waves have zonal wavelengths significantly less than the equatorial radius of deformation $a = (c/\beta)^{1/2}$. For the first baroclinic mode this parameter is about $300km$ so on a global scale their wavelength is very small. Their principal application is in the study of meridional jets which form along western boundaries in both the ocean and the atmosphere. When long wave Rossby waves strike a western boundary they reflect as Kelvin waves (see below) and as the short wave Rossby waves which are generally dispersed due to horizontal turbulence before they can propagate far to the east. This is the reason for their main significance for western boundary currents.

Another very important property of Rossby waves is that they have a maximum frequency. This may be obtained by finding the turning point of ω in relation (10):

$$\omega_{\max}^2 = \frac{\beta c}{4(2n+1)}$$

For the gravest Rossby mode ($n = 1$) and the first baroclinic vertical mode in the ocean, this corresponds to a period of oscillation of around 15 days. Forcing of greater frequency than this will not excite Rossby waves and instead excites Kelvin and gravity waves (see below). Note also that if the forcing has a particular spatial scale (i.e. a fixed value for k) then this maximum frequency will be lower. Note also that the higher the value of n the lower this frequency as well.

3 Kelvin waves

These waves have no meridional velocity and satisfy the nondispersive relation

$$\omega = kc$$

which means they propagate towards the east with the shallow water speed. Their zonal velocity and h structure is given by simply substituting $V = 0$ into (1) and using (3):

$$\begin{aligned} U &= U_0 \exp(ikx - i\omega t)\, D_0(\xi) \\ h &= cU_0 \exp(ikx - i\omega t)\, D_0(\xi) \end{aligned}$$

There now exist very good observations along the equator and these waves can be clearly seen propagating most typically with the first baroclinic shallow water speed. These waves, are like long wave Rossby waves, fundamental to equatorial dynamical adjustment.

4 Gravity waves

An approximate dispersion relation for these waves can be obtained by assuming ω large in the dispersion relation (5). If this is done then the third term on the left hand side may be neglected and so we obtain

$$\omega^2 = c^2 \left(k^2 + \frac{\beta}{c}(2n+1) \right) \tag{13}$$

This relation should be compared to that which was derived in the previous lecture

5 The equatorial waveguide

The fundamental differential equation (4) governing the equatorial $\beta-$plane may be rewritten as

$$V_{yy} + \frac{\beta^2}{c^2}\left(Y^2 - y^2\right) V = 0$$

where

$$Y^2 = \left(\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega} \right) \frac{c^2}{\beta^2} \tag{14}$$

Considering disturbances whose meridional length scale is sufficiently small so that y can be considered approximately constant then it is clear that these disturbances will be oscillatory for

$$y^2 < Y^2$$

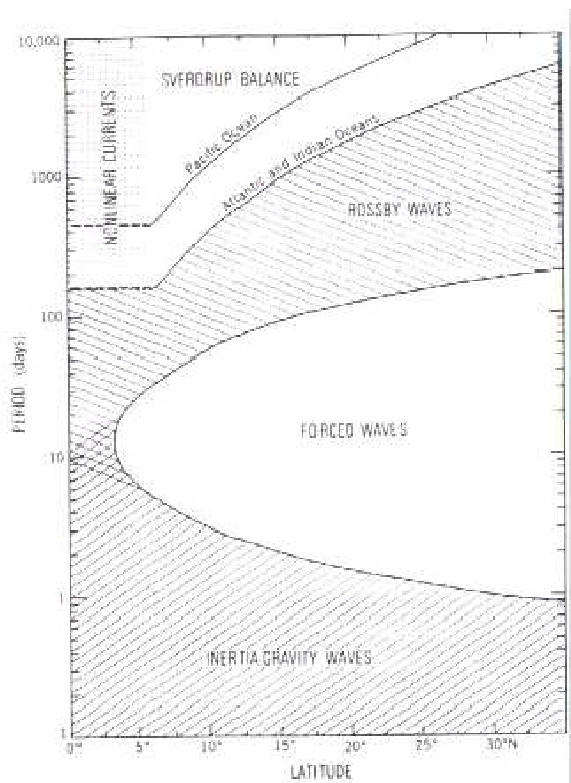
and decay with increasing y otherwise. Thus Y can be considered as measuring the width of a channel in which waves are possible. Such a channel is commonly referred to as the *equatorial waveguide.* It is clear from (14) that Y^2 will have a maximum when

$$k = \frac{\beta}{2\omega}$$

and we have

$$\mathrm{Y}_{\max} = \frac{\omega^2}{\beta^2} + \frac{c^2}{4\omega^2} \tag{15}$$

This relationship is plotted in Figure 5.



Note that as the latitude increases then the frequency range for which waves are possible widens and a large separation between gravity and Rossby waves occurs. Near the equator this separation vanishes because of the presence of the mixed Rossby-gravity wave (the solution with $n = 0$). The parabola depicting the relation (15) has a minimum value when

$$\omega = \sqrt{\frac{\beta c}{2}}$$

which indicates the centre of the frequency band at mid-latitudes for which no propagating wave-like solutions are possible. This frequency corresponds to periods of around 15 days for the first baroclinic mode.

It is interesting to consider in more detail the solutions that occur when the meridional length scale of a wave is small compared to the waveguide. The we can make the so-called WKB approximation and consider approximate solutions of (4) of the form

$$\exp\left(i(kx + ly - \omega t)\right)$$

where l is the meridional wavenumber. It follows then that

$$l^2 = \frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega} - \frac{\beta^2 y^2}{c^2} \tag{16}$$

Consider now a wave packet travelling according to this dispersion relation. It's tangent vector will be specified by the ratio of the group velocity in the zonal and meridional directions:

$$\frac{dy}{dx} = \frac{\partial \omega}{\partial l} / \frac{\partial \omega}{\partial k} = \frac{l}{k + \beta/2\omega}$$

Substituting the expression (16) into this and solving the first order ODE we get

$$y = Y \sin\left(\frac{\beta x}{c(k + \beta/2\omega)} + \theta_0\right)$$

where θ_0 is an integration constant. Thus we can see that the waves oscillate in the waveguide with a maximum y amplitude given by the waveguide width Y. Note also for long waves the wavelength is approximately only dependent on the frequency of the wave.

6 Forced equations

As we saw in the previous lecture the effects of wind stress forcing of the ocean and diabatic heating of the atmosphere can be easily taken into account in the linearized primitive equations by incorporating forcing terms into the shallow water equations which describe the horizontal flow. Thus we have

$$\begin{aligned} U_t - \beta y V &= -h_x + rX \\ V_t + \beta y U &= -h_y + rY \\ h_t + c^2(u_x + v_y) &= -sQ \end{aligned} \tag{17}$$

where (X, Y) is the wind stress forcing and Q is the diabatic heating. The coefficients r and s are determined by the projection onto the vertical modes as outlined previously.

The forced shallow water equations can be solved now by projecting the forcing onto the parabolic cylinder functions. For example we can write

$$X = \sum_{n=1}^{\infty} X_n(x,t) D_n(y)$$

In order to simplify the presentation we now introduce non-dimensionalization of our equations. Here we assume that the natural velocity scale is c and the natural length scale is the equatorial radius of deformation $a = \sqrt{c/\beta}$. Scaling our variables by these we obtain the simplified equations

$$\begin{aligned} U_t - yV &= -h_x + rX \\ V_t + yU &= -h_y + rY \\ h_t + (u_x + v_y) &= -sQ \end{aligned} \tag{18}$$

A further change of variables is convenient. We introduce the ancillary variables

$$\begin{aligned} q &= h + U \\ r &= h - U \end{aligned}$$

Adding and subtracting the first and third equations of (18) while retaining the second leads then easily to the equations

$$\begin{aligned} q_t + q_x + V_y - yV &= rX - sQ \\ r_t - r_x + V_y + yV &= -rX - sQ \\ q_y + yq + r_y - yr + 2V_t &= 2rY \end{aligned} \tag{19}$$

We now decompose all variables into their parabolic cylinder function components which we label with an integer superscript. Now a particularly useful set of relations for the parabolic cylinder functions are those of the raising and lowering operators:

$$\begin{aligned} (\partial_y + y)\, D_n(y) &= 2nD_{n-1}(y) \\ (\partial_y - y)\, D_n(y) &= -D_{n+1}(y) \end{aligned}$$

It now follows easily from (19) that we have an infinite set of constant coefficient linear equations for the variables q^n, r^n and V^n :

$$\begin{aligned} q^n_t + q^n_x - V^{n-1} &= rX^n - sQ^n \\ r^n_t - r^n_x + 2(n+1)V^{n+1} &= -rX^n - sQ^n \\ 2(n+1)q^{n+1} - r^{n-1} + 2V^n_t &= 2rY^n \end{aligned} \tag{20}$$

where $n = 0, 1, 2, 3, \ldots$ and if superscripts are negative then the term vanishes. Obviously these equations can be solved in a straightforward (albeit sometimes tedious) fashion once X^n, Y^n and Q^n are computed. We will discuss particular solutions in detail later.

7 Boundary reflections

In the ocean reflection of the waves discussed above is an important process for the dynamic adjustment of the various tropical basins (Pacific, Atlantic and Indian). At low frequencies there are three types of waves: The eastward propagating Kelvin wave which are highly equatorially trapped; the long wave Rossby waves which propagate to the west with considerably slower group velocities than the Kelvin wave and finally the short wave Rossby waves which propagate eastward. Rossby waves of both types can exist at arbitrary distance from the equator although their speed slows as latitude increases.

It is clear then that when a long wave Rossby wave impinges on a western boundary it can reflect as both a Kelvin wave and as a packet of short wave Rossby waves. Both processes can be shown analytically to occur. The first reflected wave propagates rapidly back into the interior of the basin while the second packet travels very slowly and is dissipated before it can travel very far from the boundary. The effect of this packet is to modify the western boundary current systems such as the Gulf stream in the Atlantic and the Kuroshio in the Pacific.

The other significant reflection process occurs at the eastern boundary where an impinging Kelvin wave is reflected as a packet of long wave Rossby waves. The reflected packet tends to be much less equatorially confined than the impinging Kelvin wave and a significant fraction of the incident energy travels along the boundary to high latitudes.

Both these reflection processes will be discussed in more detail in the next lecture.