Atmospheric Dynamics

Lecture 3: Circulation and Vorticity

1 Circulation

Considerations of angular momentum of fluid parcels is particularly important in understanding atmospheric dynamics. Here we introduce the basic concepts.

Circulation C is defined as the line integral about a closed contour within a fluid of the local velocity of elements

$$C = \oint \overrightarrow{U} \cdot dl$$

If the contour is taken to be a circle of radius R rotating as a solid body with angular velocity Ω then it is easily shown that

$$C = \int_0^{2\pi} \Omega R^2 d\lambda = 2\pi\Omega R^2$$

showing that the circulation is just 2π times the angular momentum of the circle. Circulation has the advantage over angular velocity that no assumption of a solid body is required and so it is suited to describing angular momentum ideas in a fluid. The time rate of change of circulation for a fluid element can easily be computed (using the equations from Lecture 1) and is quite revealing:

$$\begin{aligned} \frac{dC}{dt} &= \oint \frac{d\overrightarrow{U} \cdot dl}{dt} = \oint \left\{ \frac{d\overrightarrow{U}.}{dt} \cdot dl + \overrightarrow{U} \cdot d\overrightarrow{U} \right\} \qquad (1) \\ &= -\oint \left\{ \frac{\nabla p \cdot dl}{\rho} - \nabla\Phi \cdot dl \right\} + \frac{1}{2} \oint d\left(\overrightarrow{U} \cdot \overrightarrow{U} \right) \\ &= -\oint \rho^{-1} dp \end{aligned}$$

as the second and third closed line integrals on line 2 vanish. The final integral is called the *solenoidal* term. If the density is only a function of pressure, a situation referred to as a barotropic fluid, then the solenoidal term vanishes due to the properties of line integrals and so circulation is conserved following barotropic fluid elements.

It is also useful to apply Stokes Theorem to obtain

$$C = \oint \overrightarrow{U} \cdot dl = \int_S \int (\nabla \times \overrightarrow{U}) \cdot n d\sigma \equiv \int_S \int \overrightarrow{\varsigma} \cdot n d\sigma \qquad (2)$$

where S is a surface enclosing by the closed contour and n is a normal vector to this surface. The quantity $\overrightarrow{\varsigma}$ is referred to as the *vorticity* of the fluid and will be discussed further below.

Given that we are dealing with a rotating frame of reference (see Lecture 1) it is convenient to separate out what is known as the *relative* and *absolute* circulation and vorticity: Using the fourth equation from Lecture 1 we have

$$\overrightarrow{\varsigma}_a = \overrightarrow{\varsigma} + \nabla \times (\overrightarrow{\Omega} \times \overrightarrow{x}^r) = \overrightarrow{\varsigma} + 2\overrightarrow{\Omega}$$

where $\overrightarrow{x}^r$ is the position vector relative to the Earth's center and Ω is the Earth's rotation vector. Note that we use the subscript a to denote absolute quantities and drop subscripts when referring to relative quantities and that we have used a standard identity from vector calculus. Applying this relation to equation (2) we obtain a relationship between absolute and relative circulation:

$$C_a = \int_S \int (\overrightarrow{\varsigma} + 2\overrightarrow{\Omega}) \cdot \overrightarrow{n} \, d\sigma = C + 2 \int_S \int \overrightarrow{\Omega} \cdot \overrightarrow{n} \, d\sigma$$

If S is a (locally) horizontal plane of area A then we have

$$C_a = C + 2\Omega \sin \phi A = C + fA$$

where Ω is the magnitude of $\overrightarrow{\Omega}$, ϕ is latitude and f is the Coriolis parameter discussed in Lecture 1. Clearly conservation of (absolute) circulation (as holds, for example, for a barotropic fluid) therefore implies that fluid particles changing latitude will have their relative circulation (the circulation in the usual frame of reference) modified by the change in the Coriolis parameter or by a change in the area of the fluid element.

An interesting application of the solenoidal relation is when there are horizontal variations in atmospheric density due to density gradients. In Figure 1 we have displayed a situation that applies when sea breezes develop.

Using the ideal gas law and equation (1) we can write

$$\frac{dC_a}{dt} = -\oint RT \, d \ln p$$

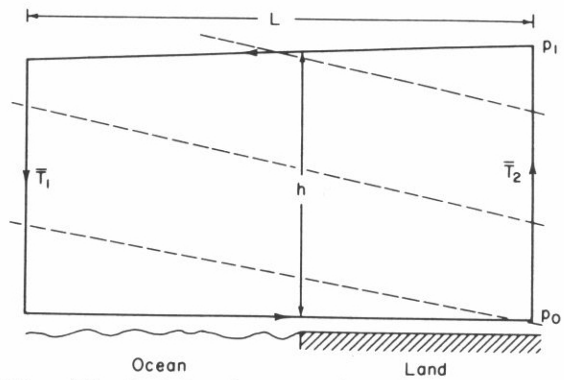
On the circuit displayed in the Figure the horizontal legs make no contribution since pressure is constant along them. Thus we may deduce that

$$\frac{dC_a}{dt} = R \ln \left(\frac{p_0}{p_1} \right) (\bar{T}_2 - \bar{T}_1) > 0$$

where the overbar indicates a weighted vertical average. This shows that there will be an acceleration in circulation about the displayed loop. This is the mechanism for the ocean sea breeze.

2 The Vorticity equation

In meteorology we are often interested in horizontal circulations such as those associated with high and low pressure vortices as will be discussed in much



Application of the circulation theorem to the sea breeze problem. The closed heavy solid line is the loop about which the circulation is to be evaluated. Dashed lines indicate surfaces of constant density.

Figure 1:

more detail in a later Lecture. As a consequence the vertical component of the vorticity vector ($\zeta \equiv v_x - u_y$) is often of considerable interest. If this is positive we have anti-clockwise flow and vice-versa for a negative value. We can obtain an equation for this by taking the (vertical part) of the curl of the momentum equations[1]:

$$\frac{\partial}{\partial x}\left(v_t + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}\right)$$
$$-\frac{\partial}{\partial y}\left(u_t + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial y}\right)$$

from which we obtain

$$\frac{d\zeta}{dt} + (\zeta + f)\,\nabla\cdot\overrightarrow{u} + \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + v\frac{\partial f}{\partial y} = \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

where the divergence is assumed two dimensional here. This may be also (more compactly) written as

$$\frac{d(\zeta + f)}{dt} + (\zeta + f)\,\nabla\cdot\overrightarrow{u} = -\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

since the Coriolis parameter f $(= 2\Omega\sin\varphi)$ only depends on y. The first term on the RHS is called the twisting term while the second is called the solenoidal term. Both are potentially sources of vorticity for a fluid parcel but for large scale flows these *source* terms for vorticity are generally small

[1]We are considering x coordinates to be in the longitudinal direction and y coordinates to be in the latitudinal direction.

as can be shown by choosing appropriate (i.e. those typical of synoptic flows) values for the variables. The details can be found in Holton on page 106-108. Generally then for such flows vorticity is generated by the presence of (horizontal) divergence. For small scale flows however the other two terms can be important- an interesting example being the first which is thought responsible for tornadoes.

3 Potential Vorticity in a shallow homogeneous layer

If we consider a fluid of this type then in the momentum equation the vertical advection terms may be neglected and horizontal pressure perturbations are due solely to variations in the fluid depth η. We may therefore write (using the hydrostatic equation)

$$u_t + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial \eta}{\partial y}$$
$$v_t + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial \eta}{\partial y}$$

It is easily verified that these equations may be rewritten as

$$\begin{aligned} u_t - (f+\zeta)v &= -B_x \\ v_t + (f+\zeta)u &= -B_y \end{aligned} \quad (3)$$

where B is the so-called Bernoulli function given by

$$B = g\eta + \frac{1}{2}(u^2+v^2)$$

Note the presence of the absolute vorticity in equations (3). We can eliminate B from these equations to obtain the relevant form of the vorticity equation for this kind of fluid:

$$\frac{d(\zeta+f)}{dt} + (\zeta+f)\,\nabla\cdot\overrightarrow{u} = 0 \quad (4)$$

which is the form of this equation that also holds approximately for large scale synoptic flow (see above). Now the equation of continuity for a shallow homogeneous layer is easily derived along the same lines as the general derivation in Lecture 1 and is

$$\frac{d(H+\eta)}{dt} + (H+\eta)\nabla\cdot\overrightarrow{u} = 0 \quad (5)$$

where H is the mean depth of the fluid (and so $H+\eta$ is the total depth of a particular column of the fluid). Equations (4) and (5) may now combined by eliminating the horizontal divergence terms. The result is that

$$\frac{dQ}{dt} = 0$$
$$Q \equiv \frac{\zeta+f}{H+\eta}$$

is called the *potential vorticity* and is conserved following fluid elements. If we consider a cylinder of this shallow fluid we can see that as this "vortex tube" is stretched vertically (and made thinner) it's absolute vorticity will need to increase in order to keep Q a constant. Note also the implications of altering latitude on the tube as well.

4 Ertel's Potential Vorticity

The equation of state for atmospheric density derived in Lecture 1 may be recast in terms of potential temperature and shows that if one considers *isentropic* surfaces (which are surfaces of constant potential temperature and entropy) then density is purely a function of pressure providing we ignore the density effects of moisture. It follows from the solenoidal theorem above that the absolute circulation on such circuits is conserved following the (adiabatic) motion of the fluid circuit. This conservation law may be written approximately as

$$\frac{d(C + f\delta A)}{dt} = 0$$

where C is the relative circulation of the parcel whose enclosed area is approximately δA. In the limit of a very small horizontal circuit we have

$$\varsigma = \lim_{\delta A \to 0} \frac{C}{\delta A}$$

by Stokes theorem. It follows therefore that the quantity

$$\delta A(\varsigma_\theta + f) \tag{6}$$

is conserved following an isentropic fluid parcel (the subscript indicating the potential temperature of the isentrope). Consider now the vortex tube displayed in Figure 2 which is confined between two isentropic surfaces which have a pressure difference δp.

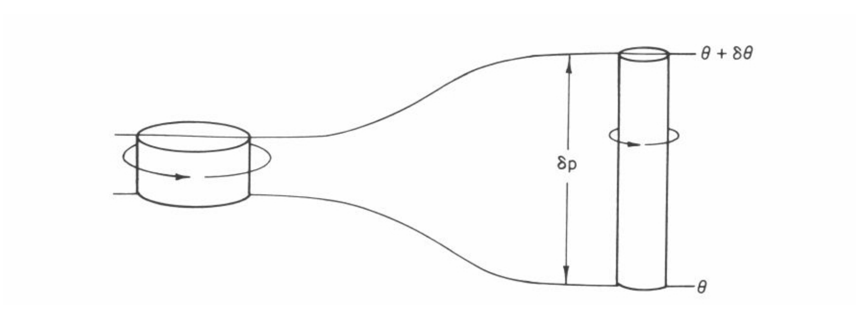


Figure 2: A cylindrical column of air moving adiabatically, conserving potential vorticity.

The mass of this parcel which must be conserved following its motion is given by

$$\delta M = \rho \delta z \delta A = -\frac{\delta p \delta A}{g}$$

from which it follows that

$$\delta A = -g\frac{\delta M}{\delta p} = g\left(-\frac{\delta\theta}{\delta p}\right)\left(\frac{\delta M}{\delta\theta}\right) = const \times g\left(-\frac{\delta\theta}{\delta p}\right) \sim K\left(-g\frac{\partial\theta}{\partial p}\right)$$

because $\delta\theta$ is a constant following the motion. It follows now from equation (6) that the quantity

$$Q = (\varsigma_\theta + f)\left(-g\frac{\partial\theta}{\partial p}\right)$$

is conserved. This quantity is known as Ertel's potential vorticity and generalizes the notion of potential vorticity developed in the previous section to a "baroclinic" environment (i.e. one in which there are significant vertical gradients in density).

The conservation of this quantity in mid-latitude flows can be used to explain why westerly flows over mountains develop downstream oscillations while easterly flows do not (see page 100-101 of Holton).