Climate Dynamics

Lecture 3: Linearization of the primitive equations and vertical normal modes

# 1 Linearization about a state of rest

Particularly useful approximate solutions of the primitive equations can be obtained by linearizing them about a state of rest and assuming that the background (mean) vertical density structure is horizontally but not vertically uniform. In order to simplify the presentation we shall assume incompressibility and hydrostatic equilibrium for the present but later consider how the general case pans out. The momentum equations for small perturbations about the state of rest are

$$\begin{array}{rcl} \frac{\partial u}{\partial t} - fv & = & -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\ \frac{\partial v}{\partial t} + fu & = & -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \\ g\rho & = & -\frac{\partial p}{\partial z} \end{array} \tag{1}$$

where we are using the abbreviations $x$ to indicate distance along circles of latitude (called the zonal coordinate), $y$ to indicate distance along longitude circles (called the meridional coordinate) and $z$ is the vertical coordinate which is perpendicular to geopotential surfaces. The Coriolis parameter $f$ is $2\Omega \sin \varphi$ and $\rho_0 = \rho_0(z)$ is the density profile. The equation of continuity in the incompressible case is

$$\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2}$$

If we assume that there is no sources of heat and salinity for the present then the incompressible version of the equation of state implies that $\frac{d\rho}{dt} = 0$ which when linearized becomes

$$\frac{\partial \rho}{\partial t} + w\frac{\partial \rho_0}{\partial z} = 0 \tag{3}$$

## 2 Separation of Variables and the vertical Sturm-Liouville equations

These five equations have the five unknowns $(u, v, w, p, \rho)$ and general solutions are possible. Those of interest to us are separable in the vertical direction. We choose the following separation for the five variables

$$\begin{aligned}
\overrightarrow{u} &= \widehat{u}(z)\overrightarrow{U}(x, y, t) \\
p &= \widehat{p}(z)\eta(x, y, t) \\
\rho &= \widehat{\rho}(z)\upsilon(x, y, t) \\
w &= \widehat{h}(z)\widetilde{w}(x, y, t)
\end{aligned}$$

where for reasons that will become clearer later we assume that $\widehat{u}$ and $\widehat{h}$ have the dimensions of length whereas $\widehat{p}$ and $\widehat{\rho}$ have their ordinary units (implying, of course, that the horizontal dependent components are dimensionless). The vertical equations can now be deduced up to a separation constant from the linearized set above. We obtain the four equations for the vertical part of the solutions:

$$\begin{aligned}
\rho_0\widehat{u} &= \widehat{p}/g \\
g\widehat{\rho} &= -\widehat{p}_z \\
\widehat{h}_z &= \widehat{u}/H_0 \\
\widehat{\rho} &= (\rho_0)_z\,\widehat{h}
\end{aligned} \tag{4}$$

where the factor $g$ has been included in the first equation for dimensional consistency and $H_0$ is the separation constant which has dimension of length. Combining the first and third equations and the second and fourth reduces this to

$$\begin{aligned}
gH_0\widehat{h}_z &= \widehat{p}/\rho_0 \\
\widehat{p}_z &= -g\,(\rho_0)_z\,\widehat{h}
\end{aligned} \tag{5}$$

and this is easily condensed further to

$$\frac{1}{\rho_0}(\rho_0\widehat{h}_z)_z + \frac{N^2}{c_e^2}\widehat{h} = 0 \tag{6}$$

where $N \equiv \left(-g\rho_0^{-1}\,(\rho_0)_z\right)^{1/2}$ is termed the Brunt-Vaisala frequency and measures the stability of the background stratification. Note that is always real if the density profile is stable. Parcels of fluid displaced within a density stratification will tend to oscillate at this frequency due to buoyancy effects. The separation constant has been "renamed" $c_e^2 \equiv gH_0$ where $c_e$ is referred to as shallow water speed for reasons that will become clearer later. The reason for assuming that $c_e^2$ is positive is because the operator $H = \frac{-1}{\rho_0 N^2}\partial_z \rho_0 \partial_z$ may easily be shown to be positive providing that the stratification is stable. To make this problem a bit more concrete let us assume that it refers to the ocean (most practical applications of this framework occur there). The lower boundary condition then is that the vertical velocity vanishes and so

$$\widehat{h}(0) = 0 \tag{7}$$

At the ocean surface the hydrostatic relation shows that the pressure at the unperturbed surface $L$ is

$$p = \rho_0 g z$$

where $z$ is the displacement. Differentiating this with respect to time gives us the upper boundary condition

$$\rho_0 g \widehat{h}(L) = \widehat{p}(L) \tag{8}$$

Equations (6), (7) and (8) together with the assumption that $N > 0$ form a Sturm-Liouville eigensystem. The mathematical literature on such systems is extensive and the eigenvalues $c_e^2$ can be shown to be discrete, bounded above, and, as already noted, positive. In the atmospheric case the upper boundary condition is at infinity and so there is a continuum part to the solution spectrum. To a high degree of accuracy in the ocean

$$\frac{-1}{\rho_0 N^2}\partial_z \rho_0 \partial_z = \frac{-1}{N^2}\partial_{zz} \tag{9}$$

and so we can see that the eigenvalues and eigenvectors are determined by the vertical profile of the Brunt-Vaisala frequency. The eigenvectors in this system are called the *normal, vertical or baroclinic/barotropic modes.* The applications of these modes are very extensive particularly in ocean dynamics. The mode with the greatest eigenvalue and the simplest (one signed) vertical structure for its eigenvector is the so-called *barotropic* mode which has a shallow water speed of around $200 - 300ms^{-1}$. The other modes are called the *baroclinic* modes and have more complex vertical structures. For the observed stratifications they have much smaller shallow water speeds (the first baroclinic mode has a typical shallow water speed of around $3ms^{-1}$).

Because the system here is Sturm-Liouville, the vertical modes satisfy an orthogonality condition. This can be obtained by subtracting two eigensolutions to equation (6):

$$\frac{d}{dz}\left(\rho_0\widehat{h}_m\frac{d\widehat{h}_n}{dz}-\rho_0\widehat{h}_n\frac{d\widehat{h}_m}{dz}\right)=\left(\frac{1}{c_m^2}-\frac{1}{c_n^2}\right)\rho_0 N^2\widehat{h}_m\widehat{h}_n$$

If this is integrated from the bottom of the ocean to the top we obtain with the use of the boundary conditions

$$\int_0^L \rho_0 N^2\widehat{h}_m(z)\widehat{h}_n(z)dz+\rho_0 g\widehat{h}_m(L)\widehat{h}_n(L)=0 \quad if \quad m\neq n \tag{10}$$

From this, the equations (5) and the boundary conditions we can also obtain

$$\int_0^L \rho_0^{-1}\widehat{p}_m(z)\widehat{p}_n(z)dz=0 \quad if \quad m\neq n \tag{11}$$

# 3 An illustrative solution to the vertical equations

To illustrate this system we consider the simplest example in which $N^2 = K^2 = const$. To further simplify matters we will assume that equation (9) holds. The solutions then of equation (6) are obviously sinusoids and cosines however boundary condition (7) means that only the former are appropriate. We write a general solution then as

$$\widehat{h}(z)=h_0\sin\lambda z$$

subject to the constraint that

$$c_n^2=K^2/\lambda^2 \tag{12}$$

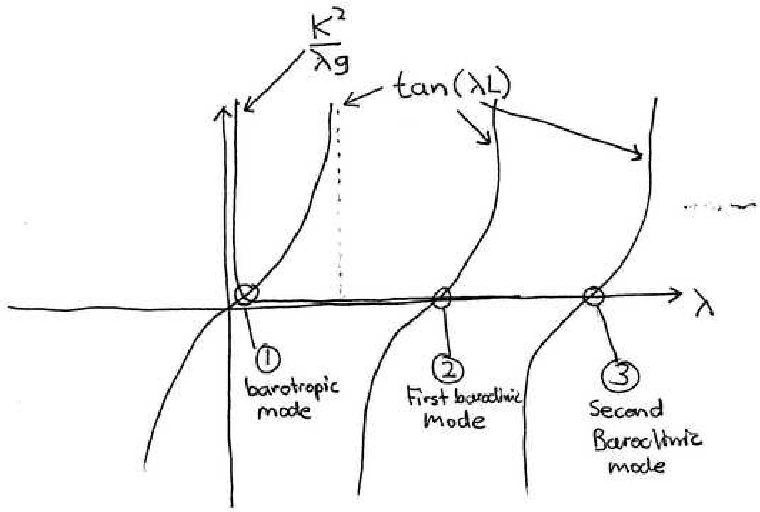
The second boundary condition (8) when combined with the first equation from (5) then leads to

$$c_n^2\lambda\cos(\lambda L)=g\sin(\lambda L) \tag{13}$$

and then combining (12) with (13) leads to

$$\tan(\lambda L)=\frac{K^2}{\lambda g} \tag{14}$$

which clearly has an infinite set of solutions which are the intersections of the hyperbola $\frac{K^2}{\lambda g}$ with the multiple branch function $\tan(\lambda L)$ (see Figure 1).



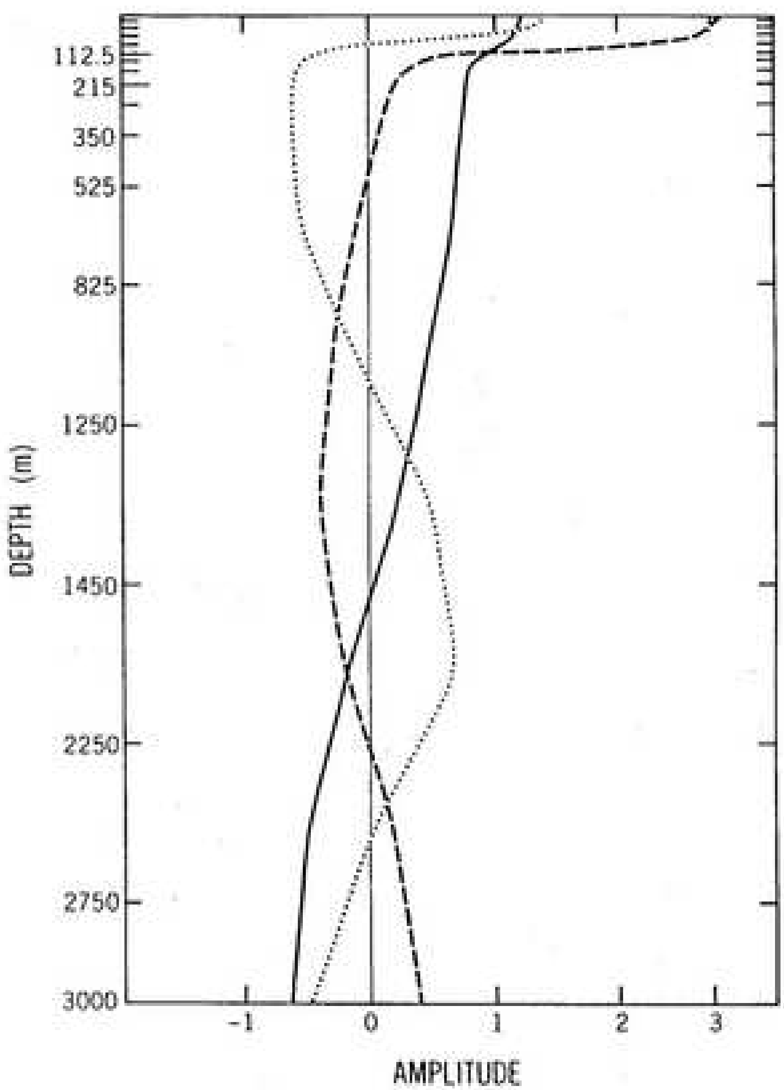

Now in general in the ocean $K \sim 10^{-3}$ which means the first solution occurs in a region for which $\tan(\lambda L) \simeq \lambda L$ and other solutions occur for $\tan(\lambda L) \simeq 0$. Thus the first (barotropic) eigenvalue is approximately

$$\begin{aligned} \lambda_0 &= \frac{K}{\sqrt{gL}} \\ c_0 &= \sqrt{gL} \end{aligned}$$

For an ocean depth of $L = 4000m$ and $K \sim 10^{-3}$ this shows that $c_0 \simeq 200ms^{-1}$. Also note that $\lambda_0 << L^{-1}$ which shows that the eigenmode has an approximately uniform structure in the vertical. For the other solutions $\lambda L \simeq 0$ implies that $\lambda_n \simeq \frac{n\pi}{L}$ and hence that

$$c_n = \frac{KL}{n\pi} \simeq \frac{1.3}{n} ms^{-1}$$

Note the $1/n$ dependency of the spectrum for higher order modes. This kind of relation is approximately true for real profiles of $N^2$. Note also that the $n'th$ baroclinic eigenmodes are sinusoids with $n$ nodes and have a vertical velocity at the surface of approximately zero. These properties hold for the "real" modes and the latter is sometimes referred to as the rigid lid approximation (it obviously cannot hold for the barotropic mode). For real profiles the first baroclinic mode is somewhat faster and many of the higher order modes have most of their structure near the surface (see Figure 2).




It is also worth noting that because the baroclinic modes have most of their vertical motion in the interior of the ocean they are sometimes called internal waves while the barotropic mode is sometimes called the exterior mode.

# 4 The shallow water equations

Corresponding to the four equations in the vertical (4) there are a set of equations governing the horizontal flow

$$\begin{array}{rcl} \frac{\partial U}{\partial t} - fV & = & -g\frac{\partial \eta}{\partial x} \\ \frac{\partial V}{\partial t} - fU & = & -g\frac{\partial \eta}{\partial y} \\ \widetilde{w} & = & -H_0(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}) \\ \widetilde{w} & = & \frac{\partial v}{\partial t} = \frac{\partial \eta}{\partial t} \end{array} \tag{15}$$

The third and fourth equations here can be combined and a new variable $h \equiv g\eta$ introduced. The resulting equations

$$\begin{array}{rcl} U_t - fV & = & -h_x \\ V_t - fU & = & -h_y \\ h_t + c_n^2(u_x + v_y) & = & 0 \end{array} \tag{16}$$

are commonly referred to as the shallow water equations. We shall consider their solution further in Lecture 4 however it is worth examining here the solutions which occur when the Coriolis term is dropped as these have wide applicability in atmospheric flows particularly and illustrate simply some important dynamical properties. The equations in this case are easily reduced to one constant coefficient linear PDE for $h$. The $x$ derivative of the first equation; the $y$ derivative of the second and the $t$ derivative of the third are all combined to obtain

$$h_{tt} = c_n^2(h_{xx} + h_{yy}) \tag{17}$$

Consider now a general wave like solution of the form

$$h = h_0 \exp(i(\omega t - kx - ly)$$

Substitution into (17) gives the wave dispersion relation

$$\omega^2 = c_n^2(k^2 + l^2)$$

Waves with this dispersion relation (or approximately) are called *gravity waves* and their group velocity is given by

$$\begin{array}{rcl} u_g = \frac{\partial \omega}{\partial k} & = & \frac{c_n^2 k}{\sqrt{k^2+l^2}} \\ v_g = \frac{\partial \omega}{\partial l} & = & \frac{c_n^2 l}{\sqrt{k^2+l^2}} \end{array}$$

which has magnitude equal to the shallow water speed for the particular vertical mode they are derived from. The direction of propagation is preferentially in the direction of smallest wavelength. Gravity waves are often seen in the atmosphere and are generated by rapidly varying forcing such as convective systems and flow over topography.

# 5 Forced Equations

The forced solutions to the linearized separable equations we have discussed have been used extensively in order to understand the atmosphere's response to heating and the ocean's response to wind forcing. The machinery for doing this is now briefly outlined.

## 5.1 Buoyancy Forcing

The most important example of this is convective heating in the atmosphere. Another application is a freshwater flux at the ocean surface. Recall that the equation of state is

$$\rho = \rho(\gamma, T, p)$$

which may be recast easily in terms of potential temperature $\theta$

$$\rho = \rho(\gamma, \theta, p)$$

Applying the chain rule leads to

$$\frac{d\rho}{dt} = \frac{1}{c_s^2}\frac{dp}{dt} + \left(\frac{\partial\rho}{\partial\gamma}\right)\frac{d\gamma}{dt} + \left(\frac{\partial\rho}{\partial\theta}\right)\frac{d\theta}{dt}$$

where $c_s$ is the speed of sound. We consider now the case of heating (the tracer forcing case is identical in spirit) for which we recall

$$\frac{dQ}{dt} = \frac{c_p T}{\theta}\frac{d\theta}{dt}$$

where $\frac{dQ}{dt}$ is the diabatic heating rate. Consider the case of incompressibility and assume there are no tracer sources then we have

$$\frac{d\rho}{dt} = -\frac{\alpha' \theta}{c_p T}\frac{dQ}{dt} \equiv -D$$

where $\alpha^{'}$ is the potential temperature expansion coefficient $\frac{1}{\rho}\frac{\partial \rho}{\partial \theta}$ which is approximately $\theta^{-1}$ for the atmosphere. Linearizing about the state of rest and mean density $\rho_0(z)$ leads to the forced version of

$$\frac{\partial \rho}{\partial t} + w\frac{\partial \rho_0}{\partial z} = -D$$

In order to solve the forced equations it is natural then to expand $D$ in terms of the vertical eigenfunctions for density. Using the final equation of (4) this expansion may be written as

$$D(x, y, z, t) = \sum_{n=0}^{\infty} (\rho_0)_z \widehat{h}_n(z) d_n(x, y, t)$$

where the $d_n$ can be obtained, up to a normalization coefficient, by use of the orthogonality properties such as those discussed above. In general, the greater the match of the heating with the particular normal mode for vertical velocity, the greater the projection coefficient $d_n$.

Solutions are then obtained by solving the forced shallow water equations i.e. where the third of the equations above becomes

$$h_t + c_n^2(u_x + v_y) \quad = \quad -d_n$$

We will discuss solutions of these equations further in a later lecture as a simple model of the tropical atmospheric circulation.

## 5.2 Windstress Forcing

From Lecture 2 it is clear that the forced linearized momentum equations may be written as

$$\begin{aligned} \frac{\partial u}{\partial t} - fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{1}{\rho_0}\frac{\partial X}{\partial z} \\ \frac{\partial v}{\partial t} + fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \frac{1}{\rho_0}\frac{\partial Y}{\partial z} \end{aligned}$$

where $(X, Y)$ is the wind stress vector. For the ocean let us assume that the atmospheric stress is deposited into the ocean in an infinitesimal layer of thickness $dz$ near the surface. Actually most is deposited in what is called the mixed layer and we will consider this somewhat more sophisticated model in a later lecture. The basic results however do not change qualitatively. Our momentum equations then become

$$\begin{aligned} \frac{\partial u}{\partial t} - fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{1}{\rho_0}X(x, y, t)\delta(z - L) \\ \frac{\partial v}{\partial t} + fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \frac{1}{\rho_0}Y(x, y, t)\delta(z - L) \end{aligned}$$

If we expand these equations into vertical modes using the $\widehat{p}_n$ then we obtain for the horizontal components

$$\begin{aligned} \tfrac{\partial U}{\partial t} - fV &= -g\tfrac{\partial \eta}{\partial x} + \tfrac{1}{\rho_0(L)} X \widehat{p}_n(L) \\ \tfrac{\partial V}{\partial t} - fU &= -g\tfrac{\partial \eta}{\partial y} + \tfrac{1}{\rho_0(L)} Y \widehat{p}_n(L) \end{aligned}$$

The important thing to notice here is that the forcing projects onto the various horizontal equations with a coefficient of $\widehat{p}_n(L)$. As we shall see later the value of this coefficient shows quite a strong dependency on stratification structure and so the windstress from the atmosphere excites different vertical modes preferentially in different geographical locations. In the western Pacific the first fastest baroclinic mode is most stimulated. In the eastern Pacific and Atlantic the second baroclinic mode is more stimulated. As these modes have quite different shallow water speeds the dynamical implications can be important.

# 6 Generalization to a compressible fluid

In this case pressure changes can cause density perturbations. The time derivative form of the equation of state (again without sources) becomes

$$\frac{d\rho}{dt} = \frac{1}{c_s^2}\frac{dp}{dt}$$

where $c_s$ is the speed of sound. The general compressible expression for the equation of mass continuity is

$$\frac{1}{c_s^2}\frac{dp}{dt} + \rho \nabla \cdot \overrightarrow{u} = 0$$

Notice that the incompressible equations can be recovered as $c_s^2 \to \infty$. Linearizing these equations about a state of rest as before we obtain the two generalizations of equations (3) and (2) for the compressible case. The density equation becomes.

$$\frac{\partial \rho}{\partial t} + w\frac{\partial \rho_0}{\partial z} = \frac{1}{c_s^2}\left(\frac{\partial p}{\partial t} - g\rho_0 w\right) \tag{18}$$

where we have used the hydrostatic equation to obtain the final term. This may be recast as

$$\frac{\partial \rho}{\partial t} - g^{-1}\rho_0 N^2 w = \frac{1}{c_s^2}\frac{\partial p}{\partial t}$$

where the Brunt Vaisala frequency for a compressible fluid has the more general form

$$N^2 = -g\left(\rho_0^{-1}\frac{\partial \rho_0}{\partial z} + \frac{g}{c_s^2}\right)$$

The continuity equation takes the form

$$\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{c_s^2}\left(\frac{\partial p}{\partial t} - g\rho_0 w\right) = 0$$

The separation of variables in the vertical proceeds as before and the third and fourth equations from (4) generalize to

$$\widehat{h}_z = \widehat{u}/H_0 - \frac{\widehat{p}}{g\rho_0 c_s^2} + \frac{g}{c_s^2}\widehat{h}$$

$$\widehat{\rho} + g^{-1}\rho_0 N^2 \widehat{h} = \frac{\widehat{p}}{c_s^2}$$

The new set of equations can then be reduced to two equations involving $\widehat{h}$ and $\widehat{p}$

$$\widehat{h}_z = \frac{1}{g\rho_0}\left[\frac{1}{c_e^2} - \frac{1}{c_s^2}\right]\widehat{p} + \frac{g}{c_s^2}\widehat{h}$$

$$\widehat{p}_z + \frac{g}{c_s^2}\widehat{p} = \rho_0 N^2 \widehat{h}$$

These equations have two solutions. The first is identical with the second order PDE applying to the incompressible case (but, of course, with the newly defined Brunt-Vaisala frequency). The second is the solution with $\widehat{h} = 0$. In this case we also have

$$\widehat{p}_z + \frac{g}{c_s^2}\widehat{p} = 0$$

and $c_e^2 = c_s^2$. In other words this solution has a shallow water speed identical to the speed of sound no associated vertical velocity and an approximate exponential fall off with height of the pressure. This additional solution is a totally compressible and is called the Lamb wave. It is important in tidal studies.

# 7 Other linearizations of the equations

We consider first some basic concepts in dynamical systems theory. A general (unforced) linearized dynamical system may be written as

$$\frac{\partial \psi}{\partial t} = A\psi \tag{19}$$

where the vector $\psi$ specifies the state of the system and the operator $A$ governs the time evolution. The eigenvectors of $A$ are often referred to as the *normal modes* and the corresponding complex eigenvalues determine the growth/decay and oscillatory frequency of these modes. In conservative dynamical systems $A$ satisfies the antihermitian property[1]

$$A = -A^*$$

where the star indicates the Hermitian conjugate. This relation implies that it has purely imaginary eigenvalues[2] which means that the normal modes oscillate with constant amplitude. This can be shown also by defining the so-called *propagator* $U(t', t)$ which takes a state vector at a time $t$ and transforms it to the appropriate state vector at time $t'$. It is easily demonstrated by discretizing (19) and iterating that

$$U(t', t) = \exp((t' - t)A)$$

from which it is easily demonstrated that if $A$ is antihermitian then $U(t', t)$ is orthogonal which implies that it preserves the norm of the state vectors.

The shallow water equations control the dynamics of solutions obtained by linearizing the primitive equations about a state of rest and we can write these in the matrix form

$$\frac{\partial}{\partial t}\begin{pmatrix} U/c \\ V/c \\ h/c^2 \end{pmatrix} = \begin{pmatrix} 0 & f & -c\frac{\partial}{\partial x} \\ -f & 0 & -c\frac{\partial}{\partial y} \\ -c\frac{\partial}{\partial x} & -c\frac{\partial}{\partial y} & 0 \end{pmatrix}\begin{pmatrix} U/c \\ V/c \\ h/c^2 \end{pmatrix}$$

where for convenience we have "non-dimensionalized" the equations using the shallow water speed. Now as is well known (and can be shown intuitively

[1] One may need to transform state variables to make this apparent.

[2] As should be well known, Hermitean operators have real eigenvalues

by discretization) the partial differential operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are antihermitian and so it follows easily that the operator $A$ for the shallow water equations is also. Thus these equations have purely oscillatory solutions and the system is conservative.

If the equations are linearized about a state of non-rest then, in general, the operator $A$ *is no longer antihermitian.* An obvious example is a linearization about a linearly varying zonal velocity $\overline{U}$. If we assume that this variation occurs in the meridional direction then

$$A = \begin{pmatrix} -\overline{U}\frac{\partial}{\partial x} & f - \overline{U}_y & -c\frac{\partial}{\partial x} \\ -f & -\overline{U}\frac{\partial}{\partial x} & -c\frac{\partial}{\partial y} \\ -c\frac{\partial}{\partial x} & -c\frac{\partial}{\partial y} & 0 \end{pmatrix}$$

In this case $A$ is obviously not antihermitian and in fact for $\overline{U}_y$ large enough there exist growing normal modes. This is commonly referred to as barotropic instability and occurs frequently in many situations in both atmosphere and ocean.