Climate Dynamics

Lecture 1: The inviscid primitive equations for atmosphere and ocean

1 Momentum

Consider an infinitesimal parcel of fluid in fixed rectilinear coordinates which has velocity $\overrightarrow{u}_f$ then it's equation of motion is

$$\frac{d\overrightarrow{u}_f}{dt} = \overrightarrow{F} \tag{1}$$

where $\overrightarrow{F}$ is the resultant force per unit mass acting on the parcel. In general there are two significant inviscid contributors to $\overrightarrow{F}$ namely pressure and gravitational potential gradients. Thus we have

$$\overrightarrow{F} = -\rho^{-1}\nabla p - \nabla\Phi_g \tag{2}$$

where p is the pressure field acting on the parcel; ρ is the density of the parcel and Φ_g is the gravitational potential. Consider the usual situation in geophysics where one is in the earth's rotating frame of reference. Now suppose that relative to this new frame of reference that the fluid particle has velocity $\overrightarrow{u}_r = \frac{d\overrightarrow{x}_r}{dt}$. By standard arguments (see an elementary physics text) the acceleration relative to this rotating frame may be written as:

$$\frac{d\overrightarrow{u}_f}{dt} = \frac{d\overrightarrow{u}_r}{dt} + 2\overrightarrow{\Omega} \times \overrightarrow{u}_r + \overrightarrow{\Omega} \times \overrightarrow{\Omega} \times \overrightarrow{x}_f \tag{3}$$

The second term on the right hand side is referred to as the Coriolis acceleration while the third is the centrifugal acceleration which is small compared to gravitation (at least on the Earth!). If we combine equations (1), (2) and (3) and further drop the r subscript then we can write down the fundamental equation for momentum on a rotating frame as

$$\frac{d\overrightarrow{u}}{dt} + 2\overrightarrow{\Omega} \times \overrightarrow{u} = -\rho^{-1}\nabla p - \nabla\Phi \tag{4}$$

where Φ is the gravitational potential and is usually simply referred to as the *geopotential*. We have neglected the centrifugal force (although it could be incorporated if the definition of geopotential were modified slighly). It is usual that we orient our reference frame so that the force $\nabla\Phi$ is regarded as vertical and in fact is equal to what is called the local acceleration due to gravity g. It is important for applications of these equations to remember that the total (parcel) derivative can be expanded using the chain rule e.g.

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \tag{5}$$

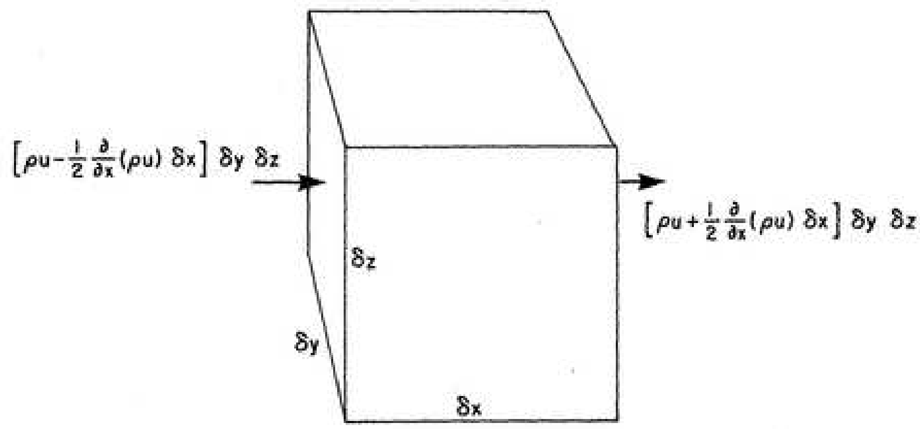
A suitable approximate coordinate system for the earth is a spherical one. In actual fact the earth is not a perfect sphere and gravity varies locally so one should really use some slight modification of oblate spheroid coordinates but the error involved is very small providing that one remembers to regard the surfaces $r = const.$ as geopotential surfaces (which aren't exactly spheres). The reason for the care needed here is that the gravitational force really dominates the equations (see the hydrostatic approximation for the vertical equation below) so unless the coordinate system is oriented properly the other horizontal coordinates will acquire large undesirable gravitational components. In terms of these new coordinates equation (4) becomesthe (complicated)

$$\begin{array}{rcl} \frac{du}{dt} - \left(2\Omega + \frac{u}{r\cos\varphi}\right)(v\sin\varphi - w\cos\varphi) & = & -\frac{1}{\rho r\cos\varphi}\frac{\partial p}{\partial\lambda} \\ \frac{dv}{dt} + \frac{wv}{r} + \left(2\Omega + \frac{u}{r\cos\varphi}\right)u\sin\varphi & = & -\frac{1}{\rho r}\frac{\partial p}{\partial\varphi} \\ \frac{dw}{dt} - \frac{v^2}{r} - \left(2\Omega + \frac{u}{r\cos\varphi}\right)u\cos\varphi & = & -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \end{array} \tag{6}$$

where r is vertical distance; λ is longitude and φ is latitude. A lot of the (messy) Coriolis terms in this equation will be negligeable for our applications (see below).

2 Density or Mass conservation

Consider a small cube in the fluid illustrated in Figure 1.



The mass flow crossing the left yz face is $\left(\rho u - \frac{1}{2}\delta x \partial(\rho u)/\partial x\right)\delta y \delta z$ whereas the flow crossing that on the right is $\left(\rho u + \frac{1}{2}\delta x \partial(\rho u)/\partial x\right)\delta y \delta z$. Summing over all faces it follows that the net change in mass for the cube is $-\nabla \cdot (\rho \overrightarrow{u})\delta x \delta y \delta z$. The rate of change per unit volume for the cube is just the rate of change of density by definition so therefore

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \overrightarrow{u}) \tag{7}$$

which is one form of the density equation. Another form can be derived by using the "advection" equation (5) as applied to ρ and combining it with (7) which leads to

$$\frac{1}{\rho}\frac{d\rho}{dt} + \nabla \cdot \overrightarrow{u} = 0 \tag{8}$$

3 Tracers

There are two tracers that are usually important in atmosphere/ocean applications namely salinity s (ocean) and moisture q (atmosphere). They are important because they affect the density of the fluid. Let γ be the mass of the tracer per unit mass of the fluid then it is obvious that the mass of the tracer per unit volume is $\rho\gamma$. So we can now apply the mass conservation equation for the whole fluid derived above to the case of mass conservation for the tracer. This is evidently done by simply replacing ρ with $\rho\gamma$ in the equations above. So we get

$$\frac{\partial(\rho\gamma)}{\partial t} = -\nabla \cdot (\rho\gamma \overrightarrow{u})$$

or

$$\frac{1}{\rho\gamma}\frac{d(\rho\gamma)}{dt} + \nabla \cdot \overrightarrow{u} = 0$$

By expanding this last expression out and using equation (8) we get

$$\frac{d\gamma}{dt} = 0 \tag{9}$$

which says that tracer concentration is conserved in a moving parcel of fluid (see however the forced equations later).

4 Equation of State

Density of both fluids depends on temperature T, pressure p and tracer concentration γ. Symbolically

$$\rho = \rho(\gamma, T, p) \tag{10}$$

In the ocean the formula is very complex and is given in Appendix 3 of Gill. It has been tested experimentally to a very high degree of accuracy. In the atmosphere the fluid is close to an ideal gas so simpler formulae are possible which are accurate to 0.1%:

$$\rho = p/(RT(1 - q + q/\epsilon))$$

where ϵ is the ratio of molecular masses of water and air (0.62197) and R is the universal gas constant.

5 Heat Equation

The first and second law of thermodynamics allows temperature T, pressure p, entropy S and heat Q to be related by the equation

$$dQ \equiv TdS = c_p dT - \frac{\alpha T}{\rho} dp \tag{11}$$

which is derived from the more basic form of the laws namely:

$$dQ = c_p dT - pd(1/\rho)$$

where c_p is the specific heat of the fluid (the rate of increase of heat content in a fluid with pressure held constant) and α is called the thermal expansion coefficient and is derivable from the equation of state using the formula

$$\alpha = -\rho^{-1}\left(\frac{\partial \rho}{\partial T}\right)_p$$

c_p is calculated experimentally in the ocean like the equation of state whereas again in the atmosphere an accurate relatively simple formula is possible

$$c_p = 1004.6(1 + 0.8375q)$$

In terms of the change of quantities in a parcel of fluid, equation (11) can be rewritten as

$$\frac{dQ}{dt} = c_p \frac{dT}{dt} - \frac{\alpha T}{\rho}\frac{dp}{dt} \tag{12}$$

which shows that temperature in a parcel of fluid can change both as a result of the input of heat and because of changes in pressure. The latter effect is much more important in the atmosphere than the ocean because the coefficient $\alpha T/\rho c_p$ is roughly a factor of 10^5 greater in the atmosphere. In the atmosphere then it is often convenient to introduce what is called *potential* temperature θ which is the temperature of a parcel of air if moved adiabatically (i.e. without adding heat) to a fixed reference pressure level. This level is usually chosen to be close to the surface such as $1000mb$. Obviously then θ only varies as a consequence of heat input. Straightforward manipulation (see Gill, Chapter 3) then allows us to write

$$\frac{dQ}{dt} = \frac{c_p T}{\theta}\frac{d\theta}{dt} \tag{13}$$

There are a number of heat sources dQ/dt for both ocean and atmosphere which we consider further in Lecture 2.

6 Common Approximations

6.1 Hydrostatic

Consider a fluid with no flow. The momentum equations show that horizontal pressure gradients must be zero while vertical pressure is given by the so-called *hydrostatic equation*

$$\frac{\partial p}{\partial z} = -\rho g \tag{14}$$

In the ocean density varies little between top and bottom which shows that pressure increases approximately linearly with depth. In the atmosphere, on the other hand, the ideal gas law holds approximately:

$$p = \rho RT$$

Combining this with (14) shows that (approximately)

$$\frac{\partial p}{\partial z} = -\frac{pg}{RT}$$

In an isothermal atmosphere this would imply an exponential drop off of pressure with height. In the real atmosphere because T also drops off with height the pressure decrease is somewhat less than exponential.

Consider now deviations from a state of rest. In general (because of mass continuity) vertical velocities are given by

$$w \sim \alpha u \equiv \frac{L_z}{L_h} u$$

where α is called the aspect ratio and L_z and L_h are the vertical and horizontal scales of interest (i.e. of the phenomenon we are trying to model). The hydrostatic equation now holds for deviations from a state of rest providing the aspect ratio is much less than 1. In a modeling context if our horizontal grid is much larger than the vertical scale of our fluid (as is most often the case in climate models) then the hydrostatic approximation is valid. There are circumstances where it isn't such as cloud resolving mesoscale atmospheric models or ocean models resolving oceanic deep convection. In both these cases the overturning cells have aspect ratios close to unity.

6.2 Incompressibility

This assumes that changes in density do not accompany changes in pressure. If the fluid has no heating or tracer source terms then the equation of state (10) and the tracer equation (9) imply that

$$\frac{d\rho}{dt} = 0$$

which means that

$$\nabla \cdot \vec{u} = 0$$

There are a number of conditions under which this latter approximation is valid:

1. Fluid velocities and wave phase speeds should be significantly less than the speed of sound. Sound waves are a compressional phenomenon and their speed $c_s = (\partial\rho/\partial p)^{-1/2}$ is around $1500ms^{-1}$ in the ocean and around $300ms^{-1}$ in the atmosphere

2. The vertical scale of motion should be significantly less than the so-called scale height of the fluid which is given by $\rho/\left|\partial\rho/\partial z\right|$ and represents roughly the height at which density falls of by a factor of e. In the ocean this height is much greater than the depth so this condition always holds. In the atmosphere the height is comparable with the height of the troposphere so for many examples here (e.g. tropical circulation) the approximation does not hold.

In general one can see that this approximation has greater uses in the ocean.

6.3 Boussinesq

This consists of ignoring variations in density in the horizontal momentum equations (those involving u and v) while taking full account of them when they have a buoyancy implication i.e. in the vertical momentum equation. This approximation is valid when the scale of motion is small compared to the scale height. Again evidently it has greater applicability in the ocean rather than the atmosphere.

6.4 Momentum equation simplification

Apart from the hydrostatic approximation, which has widespread validity in most large scale climate problems, a number of other simplifications are usually made to the spherical coordinate momentum equations (6). In general in climate applications as we have seen, vertical velocities are several orders of magnitude smaller than horizontal ones which means that terms involving them in the first two equations can be neglected. In addition the angular velocity of the earth Ω is around $10^{-4}s^{-1}$ which is generally an order of magnitude or more larger than $u/2r\cos\varphi$ since r is always close to the radius of the earth and u is never much greater than $50ms^{-1}$. Thus the first two momentum equations simplify to

$$\begin{array}{rcl} \frac{du}{dt} - 2\Omega v \sin\varphi & = & -\frac{1}{\rho r\cos\varphi}\frac{\partial p}{\partial\lambda} \\ \frac{dv}{dt} + 2\Omega u \sin\varphi & = & -\frac{1}{\rho r}\frac{\partial p}{\partial\varphi} \end{array}$$

The term $f \equiv 2\Omega\sin\varphi$ is referred to as the Coriolis parameter.

7 Conservation laws

7.1 Energy

The equation for kinetic energy of a fluid parcel of unit volume can be obtained from the momentum equation (4) by multiplying through by ρ and taking the inner product with $\overrightarrow{u}$

$$\rho\frac{d\left(\frac{1}{2}u^2\right)}{dt} = -wg\rho - \overrightarrow{u}\cdot\nabla p \tag{15}$$

where $u^2 \equiv \overrightarrow{u}\cdot\overrightarrow{u}$ and where we are assuming that the vertical velocity is defined as perpendicular to geopotential surfaces. The source terms on the right hand side represent conversion from gravitational potential energy (first term) and conversion from internal energy and the rate of working of the pressure field (second term- see below).

The internal energy equation for fluids is a variant of the heat equation introduced earlier. This is written

$$\frac{dE}{dt} = \frac{dQ}{dt} - p\nabla.\overrightarrow{u} \tag{16}$$

where the second term on the right hand side is the conversion of kinetic energy and is derived from the second heat equation and mass conservation.

The gravitational potential energy per unit mass is the geopotential Φ introduced in the momentum equation section. It depends only on z the vertical coordinate so using equation (5) gives

$$\rho \frac{d\Phi}{dt} = \rho \overrightarrow{u} \cdot \nabla\Phi = \rho \overrightarrow{u} \cdot \overrightarrow{g} = \rho w g \tag{17}$$

and we can see the kinetic energy conversion term on the right hand side.

Summing all three equations for the different kinds of energy gives the total energy equation

$$\rho \frac{d(E + \frac{1}{2}u^2 + \Phi)}{dt} = -\nabla \cdot (p \overrightarrow{u}) + \frac{dQ}{dt} \tag{18}$$

where the first right hand source term is the rate of working of the pressure field on the fluid parcel.