Mathematical modeling.

Final assignment, take home exam, due May 5 at 6pm.

1. Estimate the x value so that $x + 1/x = 5$ to within 1%. Do all arithmetic by hand. Explain your strategy.

2. A ball under the influence of gravity has $\ddot{h} = -g$ where $h(t)$ is the height of the ball over the table and $g = 9.8m/sec^2$ is the gravitational constant.

a. Check that as long as the ball is above the table (i.e., between bounces) its total energy, $E = \frac{1}{2}\dot{h}^2 + gh$, is constant.

b. Suppose I drop the ball onto the table and it bounces up with energy E_0. Find a formula for the maximum height and the time until the next bounce.

c. Suppose that the first bounce goes $2ft.$ above the table and that the ball looses 5% of its energy each bounce. About how long will it take until the bounces are less than an inch high? I am asking for the time, not just the number of bounces.

3. There is a motor that goes faster when it gets more gas but slows down because of friction. The simplest model would be

$$\dot{v} = -fv + ag ,$$

where v is the speed of the motor, f is a friction coefficient, a is an acceleration coefficient, and g is the amount of gas it gets. We want to use "feedback" to keep the speed at around $\overline{v}$. Let us suppose that the feedback takes the form

$$\dot{g} = r\left(\overline{v} - v\right) ,$$

where r is a feedback coefficient. This is an impatient feedbacker; he turns the gas knob at a rate proportional to the deviation of the speed of the motor from its desired speed.

a. Find the values of g and v that correspond to $\dot{v} = 0$ and $\dot{g} = 0$.

b. Show that the simpler feedback $g = r\left(\overline{v} - v\right)$ does not achieve $v = \overline{v}$ in steady state.

c. Suppose that f and a are fixed but the designer gets to choose r. Find the range of r values corresponding to

(i) monotone approach to equilibrium

(ii) oscillatory approach to equilibrium

(iii) instability.

4. We have the following formulae for computing x_{n+1}, x_{n+1}, and x_{n+1} from x_n, y_n, and z_n (pay attention to the extra $-$ in the y equation):

$$
\begin{aligned}
x_{n+1} &= \frac{\frac{1}{2}x_n + y_n}{1 + z_n^2} - y_n z_n \\
y_{n+1} &= \exp\left[\frac{1}{2}x_n - y_n\right] - 1 \\
z_{n+1} &= -\frac{1}{2}z_n + \log\left[1 + x_n^2 + y_n^2\right]
\end{aligned}
$$

a. If we start with x_0, y_0, and z_0 small, do we expect the iterates x_n, y_n, and z_n to get larger or smaller as n increases? Form an expectation on the basis of an eigenvalue analysis of the linearized system.

b. Use a computer (Matlab, ...) to confirm of refute this expectation?

c. Use the linearized analysis to show that some of the x_n, y_n, and z_n converge to zero faster than others. This will involve looking at the eigenvectors corresponding to the various eigenvalues. Is this confirmed by the computation? [d.] Plot $\log(|x_n|)$ as a function of n. Why is this somewhat irregular? Can you explain the overall downward trend in a quantitative way in terms of the eigenvalues?

5. There are two bags of marbles, each with 100 marbles. In the inspected bag, all marbles weigh exactly 3g. In the uninspected bag, about 20% weigh 2.8g or less. I choose a bag at random with each equally likely to be chosen. I draw 5 marbles from that bag and find that they each weigh 3g. What is the probability that I chose the inspected bag?

6. X and Y are independent exponential random variables with rate constant 1. what is the probability that $X > 1$ given that $X + Y > 2$?

7. A double server queue has two servers serving a single queue of customers. If there are two or more customers, each server is busy serving a customer. If there is just one customer, one of the servers serves that customer while the other is idle. We mark time in multiples of a discrete increment, so we write $t = 0, 1, 2, \ldots$. In a time increment, a customer that is being served leaves with probability p, with all choices being independent. If there are two customers being served, the probability that both leave is p^2. There are only n queue slots. If a new customer arrives when there are already n customers in the system, that customer gets "bumped"; we never hear from her again. At each time increment, a new customer arrives with probability q.

a. Write out the transition matrix for the case $n = 3$. This corresponds to a 4 state Markov chain.

b. Write a Matlab program to generate the transition matrix when $n = 10$.

c. When $p = .1$, $q = .19$, and $n = 10$,what is the steady state probability of a customer getting bumped and how long does it take for this steady state value to be reached? Use Matlab to compute powers of the transition matrix.