Introduction to Mathematical Modeling, Spring 2000

Assignment 3, due February 9.

1. Here we work with Taylor series and approximate solutions. This is a typical situation in applied modeling. There is a function (here $g(t)$), that we we have a definition of but not a formula. We want to build tools that explore such functions.

a. Find the first three terms of the Taylor series of $f(t,x) = e^{t\sin(x)}$ as a function of t about $t = 0$.. Think of $\sin(x)$ as a parameter. That is, take the Taylor series of e^{at} and then set $a = \sin(x)$ when you have the series.

b. Define

$$g(t) = \int_0^{2\pi} e^{t\sin(x)}dx \ .$$

Find the Taylor series for $g(t)$ for small t up to and including the t^2 term. Do this by integrating the three terms in the Taylor series from part a.

c. For small t, is $g(t)$ greater or less than one? Does this depend on whether t is positive or negative?

d. Use the answer to part b and part c to solve one of the following two equations approximately

$$1.1 = g(t) \ , \qquad .9 = g(t) \ .$$

2. Consider the iteration

$$x_{n+1} = \frac{x_n}{2 + x_n^2} \ .$$

Use a computer to automatically generate and plot the first 20 iterates, starting with $x_0 = 1$. Make a log plot (the vertical axis on a log scale). What do you notice about this plot?

3. Consider the iteration

$$\begin{aligned} x_{n+1} &= \frac{x_n^2 + 5y_n}{4 + 5(x_n^2 + y_n^2)} \ , \\ y_{n+1} &= -\left(x_n + y_n\right) \ . \end{aligned}$$

a. Use first partial derivatives to make a linear approximation to this iteration when x_n and y_n are small. Formulate this "linearized" iteration in terms of a 2×2 matrix, A.

b. Find the eigenvalues of A.

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c. From the eigenvalue information, what do you expect solutions to look like? Do the grow of decay? Do they do so in a simple monotone way?

d. Write a program to compute the first 20 iterates, starting with $x_0 = 10^{-5}$ and $y_0 = 0$. Try various plots to see how the computed iterates look, relative to your prediction in part c.