Assignment 4.

Given November 2, due November 8.

Objective: Sampling random variables.

We often have to deal with "heavy tailed" random variables. These are random variabes whose probability density goes to zero so slowly that some moments are infinite. Heavy tails are used to model the possibility of rare but large outcomes, such as insurance disasters. The simplest heavy tailed probability densities are "Pareto tails". These have a technical definition but the main examples are power laws:

$$f(x) = \frac{1}{Z} x^{-p} \quad \text{for large } x,$$

and log modified power laws:

$$f(x) = \frac{1}{Z} x^{-p} \log(x)^{-q} \quad \text{for large } x.$$

Consider the simplest examples of these:

$$f(x) = \begin{cases} \dfrac{1}{Z} x^{-p} & \text{if } x \geq 1, \\ 0 & \text{if } x < 1, \end{cases} \tag{1}$$

and

$$f(x) = \begin{cases} \dfrac{1}{Z} x^{-p} \log(x)^{-q} & \text{if } x \geq 2, \\ 0 & \text{if } x < 2, \end{cases} \tag{2}$$

Show that a power mapping function $X = \xi^{-\alpha}$ allows us to sample power law tails of the form (1). Implement this for $p = 3$. Use a histogram to show that you are sampling the correct density. Design a rejection algorithm that samples (2) when $q > 0$. Implement and test your algorithm for $p = 3$, $q = 2$. Again use a histogram to check taht you have the right distribution.

The histogram method estimates the probability density function (PDF) of a random variable from N samples $X_1, \ldots, X_N$. Choose a bin size, Δx, make bins

$$B_k = (a + j\Delta x, a + (j+1)\Delta x) \ ,$$

and count

$$N_j = \# \{k \text{ with } X_k \in B_j.\}$$

For small

$$N_j \approx \Delta x f(a + (j + \frac{1}{2})\Delta x) N \ . \tag{3}$$

The trick is to take Δx small enough so that the approximation (3) is valid but not so small that the bin counts are too small. Large N will definitely be called for.