Assignment 6

1. Consider the change of variable $x = \log(s)$.
   (a) Show that if $X_t = \log(S_t)$, and S_t is a geometric Brownian motion, then X_t is an ordinary Brownian motion with constant drift.
   (b) Show directly using the partial differential equations that the log change of variable changes the backward equation for geometric Brownian motion into the backward equation for Brownian motion with drift.

2. Let S_t be a geometric Brownian motion with zero expected return $dS_t = \sigma S_t dW_t$. Show that S_t is a martingale (clearly) but $S_t \to 0$ as $t \to \infty$. For the second part, show that S_t is expressed as an exponential of a sum of two terms, one of order $\sqrt(t)$ and one of order t.

3. Suppose $dS_t = \mu S_t dt + \sigma S_t dW_t$. Suppose t is measured in years and define the *annualized* returns Define discrete times $t_k = k\Delta t$. Suppose t is measured in years. Define the *annualized* period k return as

$$R_k = \frac{S_{t_{k+1}} - S_{t_k}}{\Delta t\, S_{t_k}} \; .$$

   Use the Euler Maruyama approximate formula for $\Delta S_k = S_{t_{k+1}} - S_{t_k}$ to find the approximate distribution of R_k. The sample average estimator of μ over a period $[0, T]$ is

$$\widehat{\mu} = \frac{1}{n} \sum_{t_k < T} R_k \; , \qquad n = \max\left\{ k \mid t_k < T \right\} \; .$$

   Find the limiting distribution $\widehat{\mu}$ in the limit $\Delta t \to 0$. Show that for fixed T, the variance of $\widehat{\mu}$ does not go to zero as $\Delta t \to 0$.

4. Suppose $X_{1,t}$ and $X_{2,t}$ are independent Brownian motions, and that

$$R_t = \sqrt{X_{1,t}^2 + X_{2,t}^2} \; .$$

   Show that R_t is an Ito process and a Markov process, which means that R_t is a diffusion process. Find the infinitesimal mean and variance

$$a(r) = \frac{1}{\Delta t} \mathrm{E}[\, R_{\Delta t} - r \mid R_0 = r]$$
$$b^2(r) = \frac{1}{\Delta t} \mathrm{E}\Big[\, (R_{\Delta t} - r)^2 \mid R_0 = r \Big]$$

   Write an SDE that R_t satisfies.

5. Let $S_{1,t}$ and $S_{2,t}$ be distinct but correlated geometric Brownian motion processes. A "weighted portfolio" of these assets is

$$P_t = w_1 S_{1,t} + w_2 S_{2,t} \ .$$

The portfolio weights w_1 and w_2 are fixed. Decide whether P_t is an Ito process, a Markov process and/or a diffusion process. Explain your answer.