Assignment 1

Correction. Exercise 1, part (a) has been corrected. The formula for X_2 has been corrected. The incorrect X_1 on the right has been replaced with the correct $\frac{1}{4}X_1$. In Exercise 1 part (c), the normalization has been corrected to $z(x_1)$.

1. (*Multi-variate Gaussians play a central role in stochastic calculus and all of applied probability. These exercises review some important properties.*)

   Suppose Z is a two component random variable with independent Gaussian components $Z_j \sim \mathcal{N}(0,1)$. Let X be the two component random variable defined by $X_1 = 2Z_1$ and $X_2 = \frac{1}{4}X_1 + \frac{\sqrt{3}}{2}Z_2$. Let Y be the two component random variable defined by $Y_1 = \sqrt{3}Z_1 + Y_2$ and $Y_2 = Z_2$.

   (a) Show that X and Y have the same distribution. This illustrates the *correlation is not causality* paradox. The X formulas seem to say that X_1 influences X_2 but X_2 does not influence X_1. The Y formulas seem to go the other way, with Y_2 coming first. You cannot tell which story is true, if either, by looking at the random variables X and Y.

   (b) A two component random variable is *Gaussian* with mean zero if it has PDF

   $$p(x_1, x_2) = \frac{1}{z} e^{-\frac{1}{2}\left(ax_1^2 + 2bx_1x_2 + cx_2^2\right)} \ .$$

   Determine the numbers z, a, b, c explicitly for the example above.

   (c) If $x = (x_1, x_2)$ is any two component random variable with PDF $p(x)$, then the conditional density of X_2, conditioned on $X_1 = x_1$ being given, is given in terms of the joint PDF by

   $$p(\, x_2 \mid x_1) = \frac{1}{z(x_1)} p(x_1, x_2) \ .$$

   (All normalization constants are called z, but they're not all the same.) Use this formula, together with the explicit formula from part (b) to write the PDF of X_2 conditioned on $X_1 = 5$. This includes finding the normalization constant for the conditional PDF.

   (d) The conditional distribution of X_2, given that $X_1 = 5$, may be found directly from the definitions of X_1 and X_2 in terms of Z_1 and Z_2. This allows you to see that the conditional distribution of X_2 is Gaussian with a certain mean and variance. Verify that the PDF for a one component Gaussian with that mean and variance is the PDF you got from part (c).

2. In future classes, we will see that if X_t is a one component Brownian motion process, then X_t has the following properties

   - $X_0 = 0$
   - If t_k is an increasing sequence of times $0 < t_1 < \cdots < t_m$, then the m component random variable $X = (X_{t_1}, \cdots, X_{t_m})$ are an m component multivariate normal.
   - $\mathrm{E}[X_t] = 0$, $\mathrm{var}(X_t) = t$.
   - $\mathrm{cov}(X_{t_2}, X_{t_1}) = \mathrm{var}(X_{t_1})$.

   (a) Use the above properties and the properties of Gaussian random variables (correlation/independence, etc.) to show that the *increments* $Y_k = X_{t_{k+1}} - X_{t_k}$ are jointly normal and independent.

   (b) Find the conditional distribution of X_{t_2}, conditional on the values of X_{t_1} and X_{t_3}. This conditional distribution is univariate Gaussian with a mean and variance that depend on the data. Show that μ, the conditional mean of X_t is given by linear interpolation from the given data. Show that the conditional variance is maximized (for given t_1 and t_3) when t_2 is at the midpoint. Show that this conditional variance is half what it would be if we only knew X_{t_1}.

3. The *Black Scholes* theory of stock option pricing relies on the *geometric Brownian motion* model of the randomly fluctuating price of a stock

$$dS_t = \mu S_t dt + \sigma S_t dW_t \ .$$

   The parameters are *rate of return*, μ, and *volatility*, σ. The *correlation coefficient* between random variables U and V is

$$\rho = \frac{\mathrm{cov}(U,V)}{\sqrt{\mathrm{var}(U)\mathrm{var}(V)}} \ .$$

   The correlation coefficient must be between -1 and 1. The extreme values $\rho = \pm 1$ are achieved only when $U = aV + b$ for some coefficients a and b. Find a two component SDE for the two price diffusion process $X_t = (S_{1,t}.S_{2,t})$ so that the individual price processes $S_{j,t}$ are geometric Brownian motions with parameters μ_j and $\sigma_j.S_{j,t}$, but the correlation coefficient between dS_1 and dS_2 is ρ. You can do this by making the "driving" Brownian motions $W_{1,t}$ and $W_{2,t}$ be linear combinations of independent Brownian motions $\widetilde{W}_1$ and $\widetilde{W}_2$.

4. Suppose X_t is a one component diffusion described the the *Ornstein Uhlenbeck* equation

$$dX_t = -\gamma X_t dt + \sigma dW_t \ .$$

   The Euler approximation is

$$X_{n+1}^{\Delta t} = X_n^{\Delta t} - \gamma X_n^{\Delta t} \Delta t + \sqrt{\Delta t}\, \sigma Z_n \ .$$

Take $\gamma - .2$ and $\sigma = 1$, $X_0 = 0$, and simulate up to time $T = 2$. You may modify the code `PathDemo.py` for this exercise. You will find that more work is needed to make a path up to a fixed time. This exercise asks you to take Δt small and to make enough paths that the histogram accurately represents the PDF. You will find that this takes too much computer time if the time step is too small and the number of paths is too large.

Code standards. All code should be in Python. It should be good code and make good output, following the code `PathDemo.py` Some of these standards are:

- Lots of comments to make it clear what the code does
- Clear variable and procedure names
- A docstring (look up the term if necessary) for each function
- Clearly labeled figures with parameters given in the title and appropriate legends.

If you are asked to code in a technical job interview, following rules like this will make the IT people who read it like your code more.

(a) Explore the PDF of $X^{\Delta t}$ at time $T = 2$ using a histograms. Show that the $\Delta t \to 0$ limit exists but that if Δt is not small enough then the PDF of $X^{\Delta t}$ (at time $T = 2$) is a little different from the $\Delta t \to 0$ limit. Take Z_n to be independent *standard normal* random variables (this means Gaussian, mean zero, variance 1).

(b) The positive random variable T is a *rate one exponential* if its PDF is $p(t) = 0$ for $t < 0$ and

$$p(t) = e^{-t} \ , \ \text{for } t \geq 0 \ .$$

Show that $\mathrm{E}[T] = 1$ and $\mathrm{var}(T) = 1$. Let U be a random variable uniformly distributed in $[0, 1]$ (a standard uniform random variable). Show that $T = -\log(U)$ is a rate one exponential.

(c) Redo part (a) taking $Z_n = T_n - 1$, which is a different way to make a random variable with mean zero and variance 1. Show that the results with a shifted exponential (this part) and a Gaussian (part (a)) are similar but not the same if Δt is small but not too small. Show that the $\Delta t \to 0$ limit is the same.