Assignment 6, due October 28

Corrections: (none yet)

Theory questions:

1. Write a formula using differentials that expresses the statement that in a small increment of time dt, a process X_t has $\mathrm{E}[dX \mid \mathcal{F}_t] = -\gamma X_t dt$, and $\mathrm{var}(dX_t \mid \mathcal{F}_t) = \mu X_t dt$. Use dW_t to represent a random variable independent of $\mathcal{F}_t$ that has mean zero and variance dt. Do not try to solve your stochastic differential equation.

2. (*An Ito Leibnitz product formula*) Suppose $X_t = f(W_t, t)$ and $Y_t = g(W_t, t)$. Show that
$$d(X_t Y_t) = (dX_t)Y_t + X_t dY_t + (dX_t dY_t) \; , \tag{1}$$
where $(dX_t dY_t)$ is what you get by multiplying the Ito's lemma expressions for dX and dY, then keeping only the dW_t^2 part, then writing $dW_t^2 = dt$. Do this by applying Ito's lemma to the function $h(w,t) = f(w,t)g(w,t)$ and $Z_t = h(W_t, t)$. Show that this works in the example $X_t = W_t^2$, $Y_t = W_t^3$. Compute your answer both directly as $d(W_t^5)$, and indirectly using (1).

3. (*Finding cancellation*) Use an appropriate backward equation to verify that $\mathrm{E}[\cos(kW_t)] = e^{-k^2 t/2}$. To do this, you need to define a value function a formula that satisfies the PDE and final condition for this value function. Also give a direct verification as follows. Write the Gaussian probability density, $u(w,t)$, for W_t. A trick for the integral $I = \int \cos(kw) e^{-w^2/2t} dw$ is: compute $\partial_k I$, use $we^{-w^2/2t} = -t\partial_w e^{w^2/2t}$, then integrate by parts. You get $\partial_t I = (\cdots)I$. The formula predicts that $\mathrm{E}[\cos(kW_t)]$ is exponentially small when $\text{st.dev.}(W_t) = \sqrt{\mathrm{var}(W_t)} \gg 1/k$. Use the fact that $1/k$ is the length scale of the cosine function to give an explanation in that $\cos(kW_t)$ is roughly equally likely to be positive as negative.

4. (*Backward equation for Brownian motion with drift*) If W_t is a Brownian motion, then $X_t = W_t + ut$ is a Brownian motion with *drift* speed u. Use the equations $\mathrm{E}[\Delta X \mid \mathcal{F}_t] = u\Delta t$ and $\mathrm{E}\Big[(\Delta X)^2 \mid \mathcal{F}_t\Big] = \Delta t + \text{(smaller)}$ to derive the backward equation satisfied by $f(x,t) = \mathrm{E}_{x,t}[V(X_T)]$. Imitate the steps in Subsection 3.1 of the Week 6 notes that lead to (26) there. The backward equation for Brownian motion with drift includes a term that contains $\partial_x f$.

5. (*A version of the Girsanov transformation for Brownian motion with drift.*) This exercise is a change of variables in the backward equation from Exercise 4. You can find the new variables, as we do here, as a change of variables in the backward equation. We will see later that it may be derived using the Girsanov re-weighting formula. Make the substitution $f(x,t) = e^{\mu x} g(x,t)$ and calculate the PDE that g satisfies. Show that for the right value of μ, which is related to u, there is no $\partial_x g$ in the g equation. Now make the substitution $h(x,t) = e^{rt} g(x,t)$. Show that for the right value of r, which depends on u, h satisfies the backward

equation for Brownian motion without drift, which is $0 = \partial_t h + \frac{1}{2}\partial_x^2 h$. Finally, reverse the transformations to write a formula for $f(x,t)$ in terms of $h(x,t)$.

Finite difference solution of the backward equation

For this exercise, X_t is Brownian motion. We run the Brownian motion up to time T and receive reward $V(X_T)$, but only if X_t has not touched the "sides" $x = \pm a$ for $t \leq T$. The appropriate value function is defined in terms of a "hitting indicator" function H defined by

$$H(X_{[t,T]}) = \begin{cases} 1 & \text{if } |X_s| < a \text{ for all } t \leq s \leq T \\ 0 & \text{if } |X_s| = a \text{ for some } t \leq s \leq T \end{cases}$$

This means

$$f(x,t) = \mathrm{E}_{x,t}\left[V(X_T)H(X_{[t,T]})\right] \ .$$

The backward equation for this problem is

$$\partial_t f + \frac{1}{2}\partial_x^2 f = 0 \ , \tag{2}$$

with final condition

$$f(x,T) = V(x) \ ,$$

and boundary condition

$$f(\pm a, t) = 0 \ .$$

The posted code `Assignment6.R` computes a finite difference approximation to the solution using a *space step size*, Δx, and a *time step size*, Δt. There are N *grid points* in space, which are

$$x_j = -a + (j-1)\Delta x \ .$$

We take $\Delta x = 2a/(N-1)$, so that $x_1 = -a$ and $x_N = a$. The solution is computed at times $t_k = T - k\Delta t$. The approximate solution values are

$$f_{jk} \approx f(x_j, t_k) \ .$$

The code computes the values f_{jk} by *marching* from the final time $t_0 = T$ to the desired time $t_m = 0$. Once the numbers f_{jk} are known for $1 \leq j \leq N$, we use them to calculate all the $f_{j,k+1}$. This is a *time step.*

The time step formulas are based on finite difference approximations to the derivatives in the backward equation.

$$\partial_t f(x,t) \approx \frac{f(x,t) - f(x,t-\Delta t)}{\Delta t}$$
$$\partial_x^2 f(x,t) \approx \frac{f(x-\Delta x,t) - 2f(x,t) + f(x+\Delta x,t)}{\Delta x^2} \ .$$

At the grid points, we have, for example, $x_j + \Delta x = x_{j+1}$, and $t_k - \Delta t = t_{k+1}$ These formulas lead to

$$\partial_t f(x_j, t_k) \approx \frac{f_{jk} - f_{j,k+1}}{\Delta t}$$
$$\partial_x^2 f(x_j, t_k) \approx \frac{f_{j-1,k} - 2f_{jk} + f_{j+1,k}}{\Delta x^2} \; .$$

These approximations define finite difference approximation to the backward equation:

$$0 = \frac{f_{jk} - f_{j,k+1}}{\Delta t} + \frac{1}{2}\,\frac{f_{j-1,k} - 2f_{jk} + f_{j+1,k}}{\Delta x^2} \; . \tag{3}$$

To take a time step, we solve for the values at the new time level, t_{k+1}, in terms of the values at the current time level, t_k:

$$f_{j,k+1} = \frac{\Delta t}{2\Delta x^2} f_{j-1,k} + \left(1 - \frac{\Delta t}{\Delta x^2}\right) f_{jk} + \frac{\Delta t}{2\Delta x^2} f_{j+1,k} \; . \tag{4}$$

This equation has the form

$$f_{j,k+1} = c_- f_{j-1,k} + c_0 f_{jk} + c_+ f_{j+1,k} \; , \tag{5}$$

with

$$c_- = c_+ = \frac{\Delta t}{2\Delta x^2} \;\; , \quad c_0 = 1 - \frac{\Delta t}{\Delta x^2} \; .$$

You can check that $c_- + c_0 + c_+ = 1$. If $\Delta t \leq \Delta x^2$, the coefficients are non-negative. This is necessary for the method to work, for reasons left unsaid. The code takes $\Delta t = \frac{1}{2}\Delta x^2$, or slightly smaller.

Run the code as you downloaded it. You should get three pictures that illustrate aspects of the backward equation and the finite difference solution process. `Assignment6Figure1.pdf` shows that the finite difference computation gives a good approximation to a known solution. If $V(x,t) = \cos(\pi x/(2a))$, the exact solution of the backward equation is $f(x,t) = \cos(\pi x/(2a))e^{-\pi^2(T-t)/(2a)}$. (You should check that this satisfies the PDE and the boundary conditions.) `Assignment6Figure2.pdf` illustrates the convergence as $N \to \infty$, which is the same as $\Delta x \to 0$. Solutions have a limit as $\Delta x \to 0$. `Assignment6Figure2.pdf` shows the qualitative behavior of solutions. The final condition is a "sawtooth" function $V(x) = 0$ for $x < 0$, and $V(x) = 1 - x/a$ for $x > 0$. This satisfies the boundary conditions and is discontinuous at $x = 0$. For very small values of T, the solution smooths out the discontinuity but does little else. At longer times, the solution approaches the symmetric cosine profile of `Assignment6Figure1.pdf`. The pictures you get should be identical to the ones posted with the `Assignment6.R`.

1. Experiment with the value of N in the first two experiments to verify the convergence behavior. Comment on how long it takes the code to fun for large N. If the running time is a power of N, what power would that be? The relation $\Delta t = \Delta x^2$ is important for answering that question.

2. Derive a finite difference time stepping approximation for the backward equation for Brownian motion with drift of Exercise 4 that takes the form (5). Use the finite difference approximation

$$\partial_x f(x,t) \approx \frac{f(x+\Delta x,t) - f(x-\Delta x,t)}{2\Delta x}$$

to find the coefficients c_-, c_0, and c_+. Show that if Δx is small enough depending on u, and $\Delta t \leq \frac{1}{2}\Delta x^2$, then c_-, c_0, and c_+ are all positive.

3. Modify the Experiment 1 part of the Assignment6.R code to find a related exact solution of the backward equation for Brownian motion with speed u drift. Among other things, add an argument `u` to the `fds` and `cosSol` functions. You must also change the exact solution formula to the one that depends on u, which you derived in Exercise 5. Use this to check whether your finite difference approximation with the first derivative term is correct. In all cases the solution goes to zero as $T \to \infty$. Can you give an intuitive explanation for the fact that the solution goes to zero faster when u is large?

4. Repeat Experiment 2 with a u value that makes a significant difference in the solution. See that your method still converges as $\Delta x \to 0$.

5. Repeat Experiment 3 with $u > 0$ and $u < 0$. Modify the `PlotInfo = sprintf(...)` statement in the graphics part of the Experiment 3 code so that the value of u appears in the graph. Choose u values that show the peak of the sawtooth moving to the right or the left, depending on the value of u. Explain the direction in which the peak moves, depending on the sign of u and the definition of the value function. You probably should change the T values, and go from 4 to 3 values.