Assignment 2, due September 23

Corrections: (none yet)

1. (*Conditioning, etc.*) Suppose $X_1, \ldots, X_4$ are the results of 4 coin tosses. The possible results of a single toss are $X_k = \texttt{H}$ or $X_k = \texttt{T}$. (An old American coin had the head, for H, of a buffalo on one side and the tail, for T, on the other.) The probability space Ω is the space of all 16 possible sequences. The $\sigma-$ algebra $\mathcal{F}$ reflects the state where you know only the number of X_k with $X_k = \texttt{H}$, but not the values of k. For example, in $\mathcal{F}$, the outcomes $\omega = \texttt{HHTT}$ and $\omega' = \texttt{THTH}$ are indistinguishable. Consider the function $f(\omega)$ which records the first k with $X_k = \texttt{H}$. Set $f(\texttt{TTTT}) = 5$ and $f(\texttt{HTHT}) = 1$. Let $\mathcal{P}$ be the partition corresponding to $\mathcal{F}$.

   (a) Show that there are 5 elements of $\mathcal{P}$.

   (b) List the elements of $[\texttt{HHHT}] \in \mathcal{P}$.

   (c) Suppose that $P(\omega) = 1/16$ for each $\omega \in \Omega$. Calculate conditional expectation $g = E[f \mid \mathcal{F}]$. You may express this as $g(j)$, where j is the number of H tosses in $B_j \in \mathcal{P}$.

   (d) Calculate the numbers $P(B_j)$.

2. (*Urn process*) The urn process is a simple but not trivial one dimensional random walk. In later classes we will come back to it to see how it goes over to the Ornstein Uhlenbeck process in the limit $m \to \infty$, $T \to \infty$, but m and T related by a *scaling* that we will figure out.

   (a) Calculate the transition probabilities $c_i = \mathrm{P}(i \to i+1)$ and $a_i = \mathrm{P}(i \to i-1)$. Here i is the number of red balls. The formulas depend on m (the total number of balls), and p (the probability to put back a red ball).

   (b) Figure out the forward equation for $u_{n+1,i}$ in terms of $u_{n,i-1}$, $u_{n,i}$, and $u_{n,i+1}$, and the numbers a_i, b_i, and c_i from part a.

   (c) Write the equations satisfied by the steady state probabilities π_i. Show using algebra that these equations are satisfied by (possibly a small variation on)

   $$\pi_i = p^i (1-p)^{m-i} \binom{m}{i} . \qquad (1)$$

   The *binomial coefficient* is

   $$\binom{m}{i} = \frac{m!}{i!(m-i)!} .$$

Hint: you can relate neighboring binomial coefficients using reasoning such as (approximately)

$$\binom{m}{i+1} = \frac{m!}{(i+1)!(m-i-1)!} = \frac{m-i}{i+1}\binom{m}{i} .$$

(d) Give a more conceptual derivation of the solution formula (1) as follows. Imagine that when you start, all the balls in the urn are "stale". Each time you put a new ball in, that ball is "fresh". The colors on the fresh balls are independent of each other, and each fresh ball has probability p of being red. Eventually, all the balls will be fresh. When that happens, the probability distribution of the number of red balls is binomial.

(e) *Stirling's formula* is the approximation

$$n! \approx \sqrt{2\pi n}\, n^n e^{-n} = \sqrt{2\pi n}\, e^{n\log(n)-n} .$$

Use Stirling's formula (treating it as exact) to write an approximate formula for π_i when m i, and $m-i$ are all large. Write this in the form

$$\pi_i \approx \sqrt{\frac{m}{2\pi i(m-i)}}\, e^{-\phi(i,m)} .$$

Maximize ϕ over i (use calculus, differentiate with respect to i, ...). Show that you get $i_* \approx pm$, and argue that this is the right answer, using part c if necessary. Make a quadratic approximation to ϕ about i_* and use that to make a Gaussian approximation to π. Just substitute i_* into the prefactor. Do you get the same result as the CLT? Note (*not an action item*) that you find from this a *scaling* that $i - i_*$ is on the order of $\sqrt{m}$.

3. The *ansatz* method for solving equations is to guess the form of the solution, then find the precise solution by plugging your guess into the equation. It is not always satisfying, but it is great when it works. Consider a simple random walk on $\mathbb{Z}$ with transition probabilities a, b, and c independent of i.

   (a) Write the backward equation for this process.

   (b) Show that the backward equation has solutions of the form $f_{n,i} = \alpha_n + (i-\beta_n)^2$. Find the *recurrence relations* for α_n and β_n in terms of α_{n+1} and β_{n+1}.

   (c) Directly from the process, derive equations for $\mu_n = \mathrm{E}[X_n]$, and $\sigma_n^2 = \mathrm{var}(X_n)$. You may assume $\mu_0 = 0$ and $\sigma_0 = 0$.

   (d) Show that parts (b) and (c) are consistent, using the definition of the quantity $f_{n,i}$ in the backward equation.

4. (*computing*) This assignment will get you started working with `R`. I found `R` quite challenging, so be ready to be frustrated. The purpose of the assignment is to simulate the simple urn process described in the notes, see some of its properties, and check that some agree with theory.

   (a) Download the files `Assignment2.R` and `AssignmentStart2.pdf`. Put them in a convenient directory where you will put your `R` files. Open your `R` application in that directory, or, if you will use the command line, just `cd` to that directory. If you use the `R` application, type: `source("Assignment2.R")` It should create a file `Assignment2.pdf` that is identical to `AssignmentStart2.pdf`. It worked for me.

   (b) You will notice that the transition probabilities are not the ones for the urn process with p given in the file `Assignment2.R`. Put the correct formulas in and run the code again. The solution of the forward equation should still be correct. This is your first computational result. Describe how the fit gets better or worse as you increase the number of paths with other parameters staying the same. Explain why this happens.

   (c) Show that when T is large, the distribution $u_{T,i}$ is nearly equal to π_i from (1). You will need to add another line to the graph to represent π. Choose T so that you can see the difference between u_T and π. Experiment until you get a good graph, then "print".

   (d) Play with your code to see now large a T it takes for the probability distribution u_T to converge reasonably well to π. See if you can guess a scaling T_* as a power of m. This scaling is what we use when we approximate the urn process by Ornstein Uhlenbeck later.