Sample questions for the final exam.

Instructions for the final:

- The final is Monday, December 16 from 7:10 to 9pm
- Explain all answers, possibly briefly. A correct answer with no explanation may receive no credit.
- You will get 20% credit for not answering a question. Points will be subtracted if you give a wrong answer.
- Cross off anything you think is wrong. You will have points subtracted for wrong answers even if the correct answer also appears.
- You may use one 8\(\frac{1}{2}\) × 11 piece of paper, a cheat sheet with anything you want written on it.
- The actual exam will be shorter than this.

True/False. In each case, state whether the statement is true of false and give a few words or sentence of explanation.

1. If \(X_t\) is a stochastic process with \(E[(X_{t+\Delta t} - X_t)^2 | \mathcal{F}_t] = a_t \Delta t + O(\Delta t^2)\) and \(E[(X_{t+\Delta t} - X_t)^4 | \mathcal{F}_t] = o(\Delta t)\), then \(E[|X_{t+\Delta t} - X_t|^3 | \mathcal{F}_t] = o(\Delta t)\).

2. If \(dX_t = W_t^2 dt\), then
\[
\lim_{\Delta t \to 0} \sum_{t_k < t} (X_{t_{k+1}} - X_{t_k})^2 = 0 .
\]
This uses our usual notation: \(\Delta t > 0\) is the time step, \(t_k = k\Delta t\) is a discrete time, \(t > 0\) is a fixed time as \(\Delta t \to 0\).

3. If \(X_t\) is a diffusion process and a martingale, and if \(a_t\) is adapted, then
\[
Y_t = \int_0^t a_s dX_s
\]
is a martingale and a diffusion process.

4. Suppose \(P\) and \(Q\) are probability measures on a measure space \(\Omega\). If \(P\) is absolutely continuous with respect to \(Q\), then \(Q\) is absolutely continuous with respect to \(P\).

5. If \(X^m_t\) is a family of random variables with probability density \(u^m(x, t)\), and \(u^m(x, t) \to u(x, t)\) as \(m \to \infty\), then \(X^m_t \to X_t\) as \(m \to \infty\) almost surely.

Multiple choice In each case only one of the possible answers is correct. Identify the correct answer and explain why.

6. Suppose \(P\) and \(Q\) are equivalent probability measures on a measure space \(\Omega\). Which of the following is true
(a) If $A \subseteq \Omega$ is an event, then $P(A) = 0$ if and only if $Q(A) = 0$.
(b) There is a function $f(x, t)$ so that $E[V(X_T) \mid \mathcal{F}_t] = f(X_t, t)$.
(c) $f(x, t_1) = E_{x,t_1}[f(X_{t_2}, t_2)]$, where $f(x, t)$ is the function so that $E[V(X_T) \mid \mathcal{F}_t] = f(X_t, t)$.
(d) $E[V(X_T) \mid \mathcal{F}_t] = E[V(X_T) \mid \mathcal{G}_t]$.

7. Suppose

$$dX_t = V_t dt,$$
$$dV_t = -kX_t - \gamma V_t + \sigma dW_t$$

Which of the following is true?
(a) $X_t$ is a one dimensional martingale.
(b) $X_t$ is a one dimensional Markov process.
(c) $(X_t, V_t)$ is a two dimensional Markov process.
(d) $E[(X_t - \overline{X}_t)^4] = 4 \text{var}(X_t)^2$. Here $\overline{X}_t = E[X_t]$.

8. Suppose

$$dX_t = X_t Y_t dt + dW_{1,t},$$
$$dY_t = (X_t + Y_t) dt + dW_{2,t}$$

and $(X_0, Y_0) = (1, 1)$. Here, $W_{1,t}$ and $W_{2,t}$ are independent standard Brownian motions. Which PDE below is satisfied by the value function

$$f(x, y, t) = E_{x,y,t} \left[ \exp \left( - \int_t^T X_s^2 ds \right) \right].$$

(a) $\partial_t f + xy \partial_x f + (x + y) \partial_y f + \frac{1}{2} \triangle f - x^2 = 0$.
(b) $\partial_t f + xy \partial_x f + (x + y) \partial_y f + \frac{1}{2} \triangle f + x^2 f = 0$.
(c) $\partial_t f + \partial_x (xy f) + \partial_y ((x + y)f) + \frac{1}{2} \triangle f - x^2 f = 0$.
(d) $\partial_t f + xy \partial_x f + (x + y) \partial_y f + \frac{1}{2} \triangle f - x^2 f = 0$.

9. Consider the integration-by-parts formula

$$\int_0^t sX_s dX_s = \frac{1}{2} (tX_t^2 - t^2) - \frac{1}{2} \int_0^t (X_s^2 - s) ds.$$

This formula is true under which hypotheses:

(a) Only when $X_t$ is a standard Brownian motion.
(b) As long as $X_t$ is an Ito process with $dX_t = a_t dt + dW_t$.
(c) As long as $X_t$ is an Ito process.
(d) Only if \( X_t \) is a differentiable function of \( t \).

Standard long answer questions

10. Suppose \( X_{n+1} = 2X_n - X_{n-1} + Z_n \), where \( Z_n \sim \mathcal{N}(0,1) \) i.i.d. Describe a value function and backward equation that can be used to calculate \( P(X_T > 1 \mid X_0 = X_1 = 0) \).

11. An order book holds orders, which are offers to buy or sell a given stock at a given price. An order in the book can be removed, or cancelled at any time. Suppose new orders arrive to the order book as a Poisson process with rate parameter \( \lambda \). Suppose that the time until cancellation of any order is exponential with rate constant \( \mu \). Suppose all cancellations and inter-arrival times are independent. An event in the order book is an arrival of a new order or a cancellation of an existing order.

   (a) Suppose there are \( n \) orders at the book at time \( t \). What is probability distribution of the time until the next event?

   (b) What is the probability that the next event is a cancellation?

   (c) Assume there is a statistical steady state (there is). What is the expected number of orders in this steady state?

12. Suppose \( \sigma_n \) is a sequence of numbers with

\[
\sum_{n=1}^{\infty} \sigma_n^2 < \infty.
\]

Suppose that \( X_n \sim \mathcal{N}(\sigma^2) \). Do not assume that the random variables \( X_n \) are independent of each other. Show that

\[
\sum_{n=1}^{\infty} X_n^2 < \infty.
\]

13. Suppose \( 0 < T_1 < T_2 < \cdots \) are the random arrival times of a Poisson arrival process with rate constant \( \lambda \). We want to construct a re-weighting function \( L(t_1, t_2, \ldots) \) so that if \( F(t_1, t_2, \ldots) \) is a function of the arrival times, then

\[
\mathbb{E}_{\mu}[F(T_1, T_2, \ldots)] = \mathbb{E}_{\lambda}[F(T_1, T_2, \ldots)L(T_1, T_2, \ldots)].
\]

We want to simulate a Poisson arrival process with rate \( \lambda \), compute the functional \( F \), reweight with the likelihood ratio \( L \), and get the expectation with respect to the Poisson process with rate constant \( \mu \neq \lambda \). The formula for \( L \) depends on the details.

   (a) Find a formula for \( L \) if we simulate the \( \lambda \) process up to a fixed time \( t \). The number of arrivals up to time \( t \) is uncertain. Hint: Imagine that you simulate the Poisson process by first choosing the number of arrivals and then choosing the arrival times independently and uniformly in \([0, t]\), then sort them.
(b) Find the formula for $L$ if we simulate the $\lambda$ process until a fixed number $n$ of arrivals have happened. Hint: the joint PDF of the $T_k$ depends only on $T_n$ and the ordering.

14. Suppose $X_n$ is a discrete time Gaussian process $X_{n+1} = aX_n + bZ_n$, where $Z_n \sim \mathcal{N}(0,1)$ are independent for different $n$ and $X_0 = 0$. We are interested in the hitting probability

$$A = P(|X_n| \geq M \text{ for some } n \leq T).$$

Describe a Monte Carlo algorithm that evaluates $A$ by simulating the Brownian random walk $X_{n+1} = X_n + bZ_n$. This is the original process with $a = 1$. Give formulas for any quantities you need to compute, particularly likelihood ratios. Assume there is a random number generator that creates i.i.d. standard normals so that you can just use $Z_n$ in formulas.

15. Let $X_t$ be a Brownian motion with $E[(X_t - X_0)^2] = t$, starting from $X_0 = a > 0$. Let $\tau$ be the hitting time $\tau = \min \{ t \mid X_t = 0 \}$. Calculate $E[\tau^{-1}]$.

Write the probability density or distribution function then calculate the appropriate integral involving it. Hint: use the change of variable $t = s^{-2}$ to do the integral.

16. Consider the square root process $dX_t = -aX_t dt + \sigma \sqrt{X_t} dW_t$. Suppose $X_0 = 1$.

(a) Use the Ito calculus to calculate $\frac{d}{dt} E[X_t]$ and $\frac{d}{dt} E[X_t^2]$. Use these results to write a formula for $E[X_t]$ and a formula for $E[X_t^2]$. Hint: if $\dot{v} = -2av + \sigma^2 e^{-at}$, you can find $v(t)$ using the ansatz $v = \alpha e^{-at} + \beta e^{-2at}$. You find $\alpha$ from the ODE and $\beta$ from the initial condition, once $\alpha$ is known.

(b) Write the backward equation and use it to calculate $m(x,t) = E_{x,t}[X_T]$.

Assume an ansatz of the form $m(x,t) = A(t)x + B(t)$. Show that your answer is consistent with part (a).

(c) Use the backward equation to write a formula for $f(x,t) = E_{x,t}[X_T^2]$. Assume an ansatz of the form $f(x,t) = A(t)x^2 + B(t)x + C(t)$.

(d) Use some of these results to show that

$$X_t \overset{D}{\to} 0, \quad \text{as } t \to \infty.$$