Assignment 9.

Due March 29.
Problem 3 corrected March 25
and again March 27.

1. Let $X(t)$ be a Brownian motion starting at $X(0) = x_0 > 0$ and let $\tau = \min\{t \text{ with } X(t) = 0\}$.

   (a) Use the method of images (lecture 5) to find a formula for $F(t) = P(\tau > t)$ as
   $$\int_{x=0}^{\infty} u(x,t)dx \; . \tag{1}$$

   (b) Show explicitly that $F(t) \to 1$ as $t \to 0$ and $F(t) \to 0$ as $t \to \infty$.

   (c) Let $f(t)$ be the probability density for τ. We have $f(t) = -\partial_t F(t)$. Use the fact that $u(x,t)$ satisfies the heat equation to find a formula for $f(t)$ in terms of an explicit formula found from $u(x,t)$ at $x = 0$.

   (d) Show that $f(t) \to 0$ as $t \to 0$ and $t \to \infty$.

   (e) Show that $E[\tau] = \infty$. Hint: Use an approximate formula for $f(t)$ for large t based on $e^{\epsilon} \approx 1$ for small ϵ.

   (f) Calculate
   $$\int_0^{\infty} f(t)dt$$
   explicitly to check that it is a probability density. Hint: try the change of variables $y = t^{-1/2}$

   (g) If $h(x_0) = E[\tau]$ were finite, how would it depend on x_0? This is a question testing indoctrination, not mathematics.

2. Let $W \in R^n$ be a multivariate normal with mean zero and covariance matrix $C = H^{-1}$. Let $X = W + r$ with $r \in R^n$. Let $f(x)$ and $g(x)$ be the probability densities for W and X respectively. Let $L(x) = g(x)/f(x)$ be the likelihood ratio.

   (a) Find a formula of the form $L(x) = e^{(\xi \cdot x) - m}$ and identify ξ and m in terms of H or C and r.

   (b) Show that $E[V(X)] = E[V(W)L(W)]$ for any bounded function V.

3. Let $W(t) = \int_0^t Z(s)ds$ be a standard Brownian motion and define $r(t) = x_0 + at$, and let $X(t) = W(t) + r(t)$. Show that $X(t)$ is Brownian motion with *drift* in the sense that if $0 \leq t_0 < t_1 < \cdots < t_n$, then the increments $Y_k = \Delta X_k = X(t_k) - X(t_{k-1})$ are independent normals. Identify their means and variances.

4. There is a forward heat equation with drift associated to random walk with drift.

(a) Write a formula for $G(x, x_0, t)$, the probability density for $X(t)$, given that $X_0 - x_0$.

(b) Verify by direct computation of derivatives that

$$\partial_t G = \frac{1}{2}\partial_x^2 G - a\partial_x G \ .$$

(c) Suppose $X(0) = X_0$ is random with probability density $u_0(x)$. Let $u(x,t)$ be the probability density for $X(t)$. Calculate an approximation to $u(x, t + \Delta t)$ as we did in class (and in the notes) up to order Δt to find a formula for $\partial_t u(x,t)$ in terms of $\partial_x u(x,t)$ and $\partial_x^2 u(x,t)$.

5. Let $X(t)$ be a standard Brownian motion (zero drift, $x(0) = 0$) and define

$$f(x,t) = E_{x,t}\left[X(T)^2\right] \ .$$

(a) Calculate $f(x,t)$ directly using the fact that $X(T) = X(t) + \text{ Gaussian }$.

(b) Show that this function satisfies the backward equation $\partial_t f + \frac{1}{2}\partial_x^2 f$.