Assignment 8.

Due March 22.
Corrected March 21.

1. Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$. Find a formula for $E[e^X]$. Use this to find $E[e^{aX}]$ as a function of $a$, $\mu$, and $\sigma$.

2. Suppose $X = (X_1, X_2, X_3)$ is a 3 dimensional Gaussian random variable with mean zero and covariance

$$E[XX^*] = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$ 

Set $Y = X_1 + X_2 - X_3$ and $Z = 2X_1 - X_2$.

(a) Write a formula for the probability density of $Y$.

(b) Write a formula for the joint probability density for $(Y, Z)$.

(c) Find a linear combination $W = aY + bZ$ that is independent of $X_1$.

3. The Ornstein Uhlenbeck process is one of the best examples in stochastic calculus, both because it has many practical applications and because it is a simple situation in which we can calculate everything. Let $Z(t)$ be a standard white noise, and $\lambda > 0$ a positive decay rate. Define

$$X(t) = \int_{-\infty}^{t} e^{-\lambda(t-t_1)} Z(t_1) dt_1. \quad (1)$$

(a) Show that $X(t)$ informally is the solution to the differential equation

$$\frac{dX}{dt} = -\lambda X + Z(t), \quad (2)$$

and that the formula (1) gives the solution to this equation. This is informal because $X(t)$ as given by (1) is not differentiable and $Z(t)$ is not an honest function. The Ito calculus gives a more formal way to express the same thing.

(b) Use the fact that $Z(t)$ is stationary to show that $X(t)$ is stationary. This means that the statistical properties of $\tilde{Z}(t) = Z(t-t_0)$ are the same as the statistical properties of $Z(t)$, and the same for $X(t)$. Hint: change variables in the integral (1).

(c) Calculate the correlation function

$$R(s) = E[X(t)X(t+s)].$$
Hint: First show that \( R(s) = R(-s) = E[X(0)X(s)] \), using the fact that \( X \) is stationary. Then suppose \( s > 0 \) and write \( X(s) \) as an integral with respect to another variable, \( t_2 \), reverse the order of integration and use the correlation formula for white noise. Note that if \( R(s) = e^{-s} \) for \( s > 0 \), then \( R(s) = e^{-|s|} \) for all \( s \).

(d) Calculate \( \hat{R}(\xi) \), the Fourier transform of the correlation function. Hint: break the integral over \( s \) into a part with \( s > 0 \) and a part with \( s < 0 \).

(e) Find a representation of \( \tilde{X}(\xi) \) in terms of \( \tilde{Z}(\xi) \), the Fourier transform of white noise \( \tilde{X}(\xi) = A(\xi)\tilde{Z}(\xi) \). Hint: There are two ways to do this (at least). One applies the Fourier transform directly to (1) and reverses the order of integration. The other substitutes the Fourier representation of \( Z \) into (1), reverses the order of integration and recognizes the result as a Fourier representation of \( X(t) \).

(f) Show that \( |A(\xi)|^2 = \hat{R}(\xi) \), as theory says it should.

(g) Let \( \mathcal{F}_t \) be the \( \sigma \)-algebra generated by the values\(^1 \) \( X(s) \) for \( s \leq t \). Find a formula for \( E[X(t + s) \mid \mathcal{F}_t] \). Hint: Use the fact that disjoint parts of \( Z(t) \) are independent mean zero. Use this formula to give another derivation of the formula for \( R \) in part (c).

(h) The differential equation (2) suggests that \( X(t) \) is a Markov process. This means that the future depends on the past only through the present. In this case, that follows from the fact that there is a formula for \( X(t + s) \) in terms of \( X(t) \) and random variables independent of \( X(t_1) \) for \( t_1 \leq t \). Give the formula (or find it in part (g)) and explain how it implies the Markov property.

---

\(^1\)This is the same as the algebra generated by the “values” of \( Z(s) \) for \( s \leq t \).