Assignment 11.

Due April 12.
Revised (extra information in (2c) added) April 9, (2c) fixed again April 10.

1. Let X be a single component random variable with $E[X] = 0$, $E[X^2] = \sigma^2$, and $E[X^4] = \mu_4 < \infty$. Define the S_n by

$$S_n = \frac{1}{n}\sum_{k=1}^{n} X_k \ ,$$

where the X_k are independent samples of X. The *strong law* of large numbers states that $S_n \to 0$ as $n \to \infty$. Kolmogorov gave a beautiful proof assuming only that $E[|X|] < \infty$, This exercise is the first step in that direction.

(a) Show that $E[S_n^2] = \frac{1}{n}\sigma^2$.

(b) Show that

$$E[S_n^4] = 3\frac{n(n-1)}{n^4}\sigma^4 + \frac{1}{n^3}\mu^4 \ . \tag{1}$$

(c) This implies two things. First, $E[S_n^4] \approx \frac{3}{n^2}\sigma^4$ for large n. Show that this conclusion is consistent with the central limit theorem.

(d) Second, $E[S_n^4] \leq C\frac{1}{n^2}$ for all n. Use this, the fact that $\sum_{n>0}\frac{1}{n^2} < \infty$, and the Borel Cantelli type lemma (not the actual Borel Cantelli lemma, which is more complicated) of paragraph 1.13 of lecture 7 to show that $S_n \to 0$ as $n \to \infty$. Hint: if $S_n^4 \to 0$ then $S_n \to 0$ (why?).

2. Suppose $dX = a(t)dt + b(t)dW(t)$ where $a(t)$ and $b(t)$ are adapted and bounded. Assume also that for $s > 0$

$$E\left[(b(t+s) - b(t))^2 \mid \mathcal{F}_t\right] \leq Cs \ . \tag{2}$$

The *quadratic variation* of X is

$$\langle X\rangle(T) = \lim_{\Delta t\to 0}\sum_{t_k\leq T}\left(X(t_{k+1}) - X(t_k)\right)^2 \ . \tag{3}$$

Define $Y_m(T)$ to be the sum on the right side of (3) when $\Delta t = 2^{-m}$. This exercise (with slightly less than 100% rigor) shows that

$$Y_m(T) \to \int_0^T b(t)^2 dt \quad \text{as} \quad m \to \infty \ .$$

(a) Assuming a and b are bounded, use the Ito isometry formula and Cauchy Schwarz to show that if $s > 0$ then

$$E\left[(X(t+s) - X(t))^2\right] \leq Cs \; .$$

if $s \leq 1$.

(b) Use the result of part (a) to show that if $s > 0$ then

$$E\left[(X(t+s) - X(t))^2 \mid \mathcal{F}_t\right] = b^2(t)s + O(s^2) \; .$$

Hints: The main contribution comes from the bdW term, so ignore the adt at first, then figure why it contributes only to the $O(s^2)$ term. With only b, write

$$\int_t^{t+s} b(u)dW(u) = \int_t^{t+s} b(t)dW(u) + \int_t^{t+s} (b(u) - b(t))dW(u) \; .$$

The Ito isometry formula together with (2) shows that the second term is smaller than the first.

(c) Define

$$R_k = (X(t_{k+1}) - X(t_k))^2 - b(t_k)^2 \Delta t \; .$$

The difference between $Y_m(T)$ and the Riemann sum

$$\sum_{t_k \leq T} b(t_k)^2 \Delta t$$

is $S_m = \sum R_k$. Show that $E[S_m]^2 \leq CT\Delta t = CT2^{-m}$. It is difficult to show $E[(X(t_{k+1}) - X(t_k))^4 \mid \mathcal{F}_{t_k}] \leq C\Delta t^2$. You may assume this fact.

(d) Use our Borel Cantelli type argument to show that $S_m = \sum R_k \to 0$ as $m \to \infty$.

(e) Use the fact that Riemann sums converge to integrals:[1]

$$\sum_{t_k \leq T} b(t_k)^2 \Delta t \to \int_0^T b(t)^2 dt \quad \text{as} \quad m \to \infty \; .$$

to conclude that the limit in (3) satisfies

$$\langle X \rangle(T) = \int_0^T b(t)^2 dt \; .$$

3. This is an example of the Feynman–Kac formula.

(a) Consider the backward equation

$$\partial_t f + \frac{1}{2}\partial_x^2 f + \gamma x f = 0 \; . \tag{4}$$

Show that if $f(x,T) = V(x) = e^{\alpha x}$, then $f(x,t) = e^{A(t)x + B(t)}$. Hint: just plug it in and find equations for A and B. Since the solution of the backward equation is unique, this will be the one and only solution.

[1] We do not prove that here, but it is in any beginning analysis or theoretical advanced calculus book.

(b) Suppose

$$V(x) = \frac{1}{2\pi} \int e^{ipx} \widehat{V}(p) dp \; .$$

Write an integral formula for $f(x,t)$. This is the source of some of the more complicated "explicit" solutions to certain option pricing problems. These solutions are not terribly explicit because there is no formula for the Fourier integral.

(c) Write a formula for f roughly of the form

$$f(x,t) \approx E\left[\exp\left(\alpha W(T) + \gamma \int_t^T W(s)ds\right)\right] \; .$$

Note that the exponent is a Gaussian random variable with a mean and variance that depend on x and t. Use the formula for the expected value of $\exp(\mu + \sigma Z)$ to evaluate the expectation. Here, $Z \sim \mathcal{N}(0,1)$ is a standard normal. Check that the formulas still hold when the parameters are complex numbers. The answer should be the same as part (a). This uses a form of the Feynman Kac formula that is slightly more general than the version in the notes, so you might want to review the argument to check that it applies in this case. In particular, make sure you understand why $f(x,t) \to V(x)$ as $t \to T$.