Assignment 7.

Given November 4, due November 11. Last revised, November 9.
Objective: Pure and applied mathematics.

The first problems are strictly theoretical. They illustrate how clever some rigorous proofs are. The inequality (3) serves the following function: We want to understand something about the entire path F_k for $0 \leq k \leq n$. We can get bounds on F_k for particular values of k by calculating expectations (e.g. $E[F_k^2]$). Then (3) uses this to say something about the whole path. As an application, we will have an easy proof of the convergence of the approximations to the Ito integral for all $t \leq T$ once we can prove it at the single time T.

1. Let F_k be a discrete time nonnegative martingale. Let $M_n = \max_{0 \leq k \leq n} F_k$ be its *maximal function*. This problem is the proof that

$$P(M_n > f) \leq \frac{1}{f} E[F_n \mathbf{1}_{M_n \geq f}] \; . \tag{1}$$

The proof also shows that if F_k is any martingale and $M_n = \max_{0 \leq k \leq n} |F_k|$ its maximal function, then

$$P(M_n > f) \leq \frac{1}{f} E[|F_n| \, \mathbf{1}_{M_n \geq f}] \; . \tag{2}$$

These inequalities are relatives of *Markov's inequality* (also called *Chebychev's inequality*, though that term us usually applied to an interesting special case), which says that if X is a nonnegative random variable then $P(X > a) < \frac{1}{a} E[X \mathbf{1}_{X > a}]$, or, if X is any random variable, that $P(|X| > a) < \frac{1}{a} E[|X| \mathbf{1}_{|X| > a}]$.

a. Let A be the event $A = \{M_n \geq f\}$. Write A as a disjoint union of disjoint events $B_k \in \mathcal{F}_k$ so that $F_k(\omega) \geq f$ when $\omega \in B_k$. Hint: If $M_n \geq f$, there is a first k with $F_k \geq f$.

b. Since $F_k(\omega) \geq f$ for $\omega \in B_k$, show that $P(B_k) \leq \frac{1}{f} E[\mathbf{1}_{B_k} F_k]$ (the main step in the Markov/Chebychev inequality)).

c. Use the martingale property and the tower property to show that $\mathbf{1}_{B_k} F_k = E[\mathbf{1}_{B_k} F_n \mid \mathcal{F}_k]$ so $E[\mathbf{1}_{B_k} F_k] = E[\mathbf{1}_{B_k} F_n]$. Do this for discrete probability if that helps you.

d. Add these to get (1).

e. We say F_k is a *submartingale* if $G_k \leq E[G_n \mid \mathcal{F}_k]$ (warning: submartingales go up, not down). Show that if F_k is any martingale, then $|F_k|$ is a submartingale. Show (1) applies to nonnegative submartingales so (2) applies to general martingales, positive or not.

2. Let M be any nonnegative random variable. Define $\mu(f) = P(M \geq f)$, which is related to the CDF of M. Use the definition of the abstract integral to show that $E[M] = \int_0^\infty \mu(f)df$ and $E[M^2] = 2\int_0^\infty f\mu(f)df$. These formulas work even if the common value is infinite. If G is another nonnegative random variable, show that $E[GM] = \int_0^\infty E[G\mathbf{1}_{M\geq f}]df$. Of course, one way to do this is to formulate a single general formula that each of these is a special case of.

3. Use the formulas of part 2 together with Doob's inequality (2) to show that

$$E[M_n^2] \leq 2E[M_n\,|F_n|]\ ,$$

so

$$E[M_n^2] \leq 2E\left[F_n^2\right]\ . \tag{3}$$

(It will help to use the *Cauchy Schwarz inequality* $E[XY] \leq (E[X^2]E[Y^2])^{1/2}$.)

Now some more concrete examples. We can think of martingales as absolutely non mean reverting. The inequality (3) expresses that fact in one way: the maximum of a martingale is comparable to its value at the final time, on average. The Ornstein Uhlenbeck process is the simplest continuous time mean reverting process, a continuous time anologue of the simple urn model.

4. An *Ornstein Uhlenbeck* process is an adapted process $X(t)$ that satisfies the Ito differential equation

$$dX(t) = -\gamma X(t)dt + \sigma dW(t)\ . \tag{4}$$

We cannot use Ito's lemma to calculate $dX(t)$ because $X(t)$ is not a function of $W(t)$ and t alone.

a. Examine the definition of the Ito integral and verify that if $g(t)$ is a differentiable function of t, and $dX(t) = a(t)dt + b(t)dW(t)$, with a random but bounded $b(t)$, then $d(g(t)X(t)) = \dot{g}(t)X(t)dt + g(t)dX(t)$. It may be helpful to use the Ito isometry formula (paragraph 1.17 of lecture 7).

b. Bring the drift term to the left side of (4), multiply by $e^{\gamma t}$ and integrate (using part a) to get

$$X(T) = e^{-\gamma T}X(0) + \sigma\int_0^T e^{-\gamma(T-t)}dW(t)\ .$$

c. Conclude that $X(t)$ is Gaussian for any T (if $X(0)$ is) and that the probability density for $X(T)$ has a limit as $T \to \infty$. Find the limit by computing the mean and variance.

d. Contrast the large time behavior of the Ornstein Uhlembeck process with that of Brownian motion.

5. Show that $e^{ikW(t)+k^2t/2}$ is a martingale using Ito differentiation.