Assignment 6.

Given October 25 due October 31 Last revised, October 30.

Objective: More calculations involving Brownian motion

1. *The Brownian bridge construction.* The Brownian bridge construction is useful in theory and in practical computation. It gives an easy rigorous construction of Brownian motion and allows one to prove that paths are Hölder continuous. Successful low discrepency sequence quasi Monte Carlo computations also create paths using bridges.

a. Suppose $0 \leq t_1 < t_2 < t_3$ and we know the values $x_1 = X_{t_1}$ and $x_3 = X_{t_3}$, what is the conditional probability distribution of $X_2 = X_{t_2}$? Since we know the joint density of the three values (multivariate normal ...), this is simple algebra. It is simpler if you do not compute the normalization constants explicitly. You never need to compute the normalization constants as long as the distributions, and conditional distributions, are gaussian.

Choose $n = 2^L$ and $\Delta t = T/n$ and define the "diadic" points $t_{k,L} = k\Delta t$ and diadic intervals $I_{k,L} = [t_{k,L}, t_{k+1,L}]$. Note that each level L diadic interval is divided into two equal sized level $L+1$ intervals: $I_{k,L} = I_{2k,L+1} \cup I_{2k+1,L+1}$. For a path X_t, we have the level L approximation X_t^L so that on each level L diadic interval, $I_{k,L}$, X_t^L linearly interpolates the values of X_t at the end points, $X_{t_{k,L}}$ and $X_{t_{k+1,L}}$. The "hat function", $h_{k,L}(t)$ for interval $I_{k,L}$, linearly interpolates between the values $h(t_{k,L}) = 0$, $h(t_{k+1,L}) = 0$, and $h(t_{k+1/2,L}) = 1$, where $t_{k+1/2,L} = (k+1/2)\Delta t = t_{2k+1,L+1}$ is the midpoint of $I_{k,L}$. Draw a picture to see why h is called a hat function.

b. Show that

$$X_t^{L+1} = X_t^L + \sum_{k=0}^{n-1} Y_{k,L} h_{k,L}(t) \; ,$$

where the $Y_{k,l}$ are independent mean zero gaussian random variables with variance, $E[Y_{k,L}^2] = \sigma_L^2$, that depends on L but not on k. Use part a to find a forumla for σ_L^2. Draw a picture showing X_t, X_t^L, and X_t^{L+1} for $L = 2$, to illustrate the situation.

c. Let Y be a gaussian random variable with mean μ and variance σ^2. Show that, for any $\alpha \geq \sigma$, $P(|Y-\mu| > \alpha) < e^{-\alpha^2/2\sigma^2}$. Hint: this inequality is equivalent to the same inequality for a standard normal. For that, we have $P(Y > \alpha) < \frac{1}{\sqrt{2\pi}} e^{-\alpha^2/2} \int_{z>0} \exp(-\alpha z) dz$. The integral is $1/\alpha \leq 1$ for $\alpha \geq 1$, and $1/\sqrt{2\pi} < 1/2$.

d. If Y_k for $k = 0, \ldots, n-1$ are iid normals with mean zero and variance σ^2, show that $P(|Y_k| > \alpha \text{ for some } k) < ne^{-\alpha^2/2\sigma^2}$, for $\alpha > \sigma$.

e. Show that there is a $\beta > 0$ so that

$$P(\left|X_t^{L+1} - X_t^L\right| > \Delta t^{1/4} \text{ for some } t \text{ in } [0,T]) < e^{-\beta L} \; .$$

Hint: Notice that the maximum of $\left|X_t^{L+1} - X_t^L\right|$ occurs at one of the midpoints $t_{k+1/2,t}$ and therefore is equal to one of the $Y_{k,L}$.

f. (optional) If you know the Borel Cantelli lemma and have had a course in mathematical analysis, show that the sequence of continuous functions X_t^L converges in the max norm, so that the limit, X_t is a continuous function of t (almost surely).

2. *Multidimensional Brownian motion.* For this question, we will change notation from X_t to $X(t)$ so that we can use subscripts. Let $X_1(t)$, ..., $X_n(t)$ be independent standard Brownian motions. Combine them together into a vector in R^n: $X(t) = (X_1(t), \ldots, X_n(t))^*$. This vector valued random process is the "standard Brownian motion" in R^n.

a. Use the fact that the components of $X(t)$ are independent to find a formula for $u(x,t)$, the probability density for $X(t)$ by multiplying the densities of the individual components $X_k(t)$. Simplify your answer to look like $u(x,t) = C(t)\exp(.../2t)$.

b. Verify by direct calculation that this $u(x,t)$ satisfies the n dimensional heat equation $\partial_t u = \frac{1}{2} \triangle u$, where $\triangle u$, the "laplacian" of u is given by $\triangle u = \partial_{x_1}^2 u + \cdots + \partial_{x_n}^2 u$.

c. Let $f(x,t) = E_{x,t}[V(X(T))]$. Find the backward equation for f from the relationship

$$f(x,t) = E_{x,t}[f(X + \Delta X, t + \Delta t)] \; .$$

Remember that $x \in R^n$ is now a vector with n components.

d. Let A be an $n \times n$ invertable matrix and define a "correlated Brownian motion" by $Y(t) = AX(t)$. Let $v(y,t)$ be the probability density for $Y(t)$. Find a partial differential equation (a forward equation) satisfied by v of the form

$$\partial_t v = \sum_{j,k=1}^{n} b_{jk} \partial_{y_j} \partial_{y_k} v \; .$$

Express the matrix B with entries b_{jk} in terms of A.

3. Back to one dimension and X_t, we want to calculate

$$E\left[V\left(\int_0^T X_t dt\right)\right] \; .$$

In finance, this problem arises in valuing Asian style options. The backward equation approach to this is slightly more complicated than for other problems. Define $Y_t = \int_0^t X_s ds$ and $f(x,y,t) = E[V(Y_T) \mid X_t = x, \; Y_t = y]$.

a. Show that $f_t = E[V(Y_T) \mid \mathcal{F}_t]$ is actually a function of X_t and Y_t, and that this function is $f_t(X_t, Y_t) = f(X_t, Y_t, t)$ as defined above.

b. Use the relation

$$f(x, y, t) = E_{x,y,t}[f(x + \Delta X, y + \Delta y, t + \Delta t)]$$

to write a backward equation for f. This equation has second derivatives in one of the variables and first derivatives in the other two.