Assignment 8.

Given April 27, due end of finals week.

Objective: Go throught the basics of Monte Carlo

1. Write a program to generate standard normal random variables using the Box Muller algorithm. One call to your routine should make a large number of normals, say ten thousand. Verify the correctness of the program using a histogram. Choose a small but not too small Δx and let the k^{th} bin be the interval

$$B_k = \left(x_k - \frac{\Delta x}{2}, x_k + \frac{\Delta x}{2} \right) \qquad \text{where } x_k = k\Delta x .$$

Generate a large number, N, of random variables and let N_k be the number of them that land in B_k. The expected value of N_k is

$$\begin{aligned} \mathrm{E}\left(N_k\right) &= N \cdot \Pr\left(X \in B_k\right) \\ &= N \cdot \int_{B_k} \rho(x) dx \ . \end{aligned}$$

For Δx small, the integral is well enough approximated by a panel approximation, with B_k being the only panel; try the midpoint rule or 2 point Gauss quadrature. The variance of N_k is $Np(1-p)$ where $p = \Pr(X \in B_k)$. Use this to put error bars on the histogram. Choose a sensible range of k values, and N large enough so that there is relatively little error in the largest bin counts.

2. For a parameter $\beta > 0$, we want to sample the one dimensional density

$$\rho_\beta(x) = \frac{1}{Z(\beta)} e^{-\beta h(x)} \ , \quad \text{where} \quad h(x) = \frac{1}{2}\left(x^2 + x^4\right) \ .$$

Write a program to do this by rejection from a gaussian with appropriate β:

$$\rho_{0\beta}(x) = \frac{1}{Z_0(\beta)} e^{-\beta h_0(x)} \ , \quad \text{where} \quad h_0(x) = \frac{1}{2} x^2 \ .$$

Verify that the generator is correct using the histogram method from part 1. For this verification, and only for this, you need to find $Z(\beta)$. Comment on how the efficiency depends on β.

3. Now define

$$S_n = \Delta x \left(X_1 + \cdots + X_n\right) \ , \quad \beta(s) = \frac{1}{2} + s^2 \ , \quad \beta_k = \beta(S_k) \ ,$$

and choose $X_{k+1} \sim \rho_{\beta_k}$, with the convention that $\beta_0 = 1/2$. This corresponds to a rondom walk in a variable temperature environment. Estimate

$$\mathrm{E}\left(S_n^2\right) \quad \text{with } n = 100, \, \Delta x = .1.$$

Error bars are required.

4. (Extra credit, do as time permits.) We want an accurate estimate of the very small

$$\Pr\left(S_n \geq r\right)$$

with $r = 4$, $\Delta x = .1$, and $n = 100$. Do this by importance sampling, replacing $h(x)$ by

$$h_\mu(x) = \frac{1}{2}\left(x^2 + x^4 - \mu x\right) \quad .$$

Experiment with μ to find a value that gives the largest variance reduction.

5. (Time permitting but with no class credit.) Relax, you're done.