Assignment 4
Multidimensional approximation, pure theory

Given February 16, due March 2.

Local polynomial approximation approximation in more complicated in more than one dimension. Let $x = (x_1, x_2)$ be a point in two dimensional space. The unit square[1] is

$$Q = \{x \ : \ 0 \leq x_1 \leq 1 \ , \ \ 0 \leq x_2 \leq 1\} \quad .$$

We have a smooth function, f, defined on Q. We have values of f at n points[2] x_1, ..., x_n, and wish to find an "interpolating" function $\hat{f}$, to estimate f at other points inside Q. In order to do this, we will suppose that the points are the *vertices* of a *triangulation* of Q, and use piecewise linear interpolation. This is the simplest kind of interpolation in 2D, just as piecewise linear interpolation is the simplest kind of interpolation in 1D. Unfortunately, that does not make it simple.

A *triangulation* of Q is a covering of Q with a collection of non overlapping triangles. A triangle has a *boundary*, which consists of the three vertices and the three edges connecting them, and an *interior*, which is the inside excluding the boundary. The triangles in a triangulation should be non overlapping in that no point is in the interior of more than one triangle. Any point in the interior of Q and on the boundary of a triangle must be on the boundary of at least one other triangle.

Suppose that a single triangle, T, has vertices A, B, and C. We say that $\hat{f}$ is affine in T if there is a formula,

$$\hat{f}(x) = a + b_1 x_1 + b_2 x_2 \quad ,$$

which holds for all $x \in T$. The linear interpolation problem from the vertices of a triangle is to find $\hat{f}$ in T that is affine in T and satisfies the interpolation conditions:

$$\hat{f}(A) = f_A \quad , \quad \hat{f}(B) = f_B \quad , \quad \hat{f}(C) = f_C \quad , \tag{1}$$

where the values f_A, f_B, and f_C are given.

1. Show that the linear interpolation problem on any triangle is solvable whenever the area of T is not zero.

Higher order polynomial interpolation in more than one dimension gets complicated fast. The function $\hat{f}$ is quadratic in T if there is a formula,

$$\hat{f}(x) = a + b_1 x_1 + b_2 x_2 + c_0 x_1^2 + c_1 x_1 x_2 + c_2 x_2^2 \quad , \tag{2}$$

which holds for all $x \in T$. Let us denote the midpoint of the segment from A to B by $\overline{AB}$. This notation reminds us that the midpoint is the average of A and B, and is not supposed to

[1] "Q" is for Quadrat, the German word for square.

[2] It should be clear whether x_1 is the first point or the first component of a point x, any less ambiguous notation seems much more clumsy.

represent the segment from A to B. For quadratic interpolation, we need more interpolation conditions. One often used set of conditions is interpolation at the midpoints of the edges:

$$\hat{f}(\overline{AB}) = f_{\overline{AB}} \ , \quad \hat{f}(\overline{AC}) = f_{\overline{AC}} \ , \quad \hat{f}(\overline{BC}) = f_{\overline{BC}} \ , \tag{3}$$

where again the values $f_{\overline{AB}}$, $f_{\overline{AC}}$, and $f_{\overline{BC}}$ are given.

2. Show that there is a unique $\hat{f}$ that is quadratic in T and satisfies the interpolation conditions (1) and (3), as long as the area of T is not zero.

Suppose that the points $x_1, \ldots, x_n$ are all the vertices of the triangles in a triangulation of Q. We want a function, $\hat{f}$, defined in Q, that is affine in each triangle and satisfies interpolation conditions

$$\hat{f}(x_k) = f_k \quad \text{for } k = 1, \ldots, n. \tag{4}$$

3. Show that this is possible if no two vertices coincide, and if no vertex is on the segment connecting two other vertices.

To analyze the accuracy of piecewise linear interpolation[3] on a triangulation, let h be the length of the longest edge among all the edges of all the triangles in the triangulation of Q.

4. Show that if $f_k = f(x_k)$ for $k = 1, \ldots, n$, and if f is a smooth function on Q, then there is a constant, C, depending only on f (i.e., not on the triangulation or x) so that

$$\left|\hat{f}(x) - f(x)\right| \leq Ch^2 \quad \text{for all } x \in Q. \tag{5}$$

Hint: show that the Taylor series expansion of f about the center of a triangle, T, gives a linear function that is both a good approximation to f in T and almost satisfies the interpolation conditions at the vertices of T.

Now suppose that $\hat{f}$ is piecewise quadratic on a triangulation of Q and interpolates f at the vertices and at the midpoints of the edges.

5. Show that

$$\left|\hat{f}(x) - f(x)\right| \leq Ch^3 \quad \text{for all } x \in Q.$$

6. In either case (piecewise linear or quadratic interpolation), show that $\hat{f}$ is continuous. *Hint:* what this really means is that if e is a common edge of triangles T and T', then the values of $\hat{f}$ defined by interpolation on T and interpolation on T' agree on e.

7. The inequality (5) shows that $\hat{f}$ is a good approximation to f when h is small no matter what triangulation is used. In the also true that $\bigtriangledown\hat{f}$ is a good approximation to $\bigtriangledown f$ under the same circumstances? Note: we showed the answer is "yes" for one dimensional interpolation.

[3] It is common to call this piecewise linear interpolation even though it should be called piecewise affine.