Assignment 2.

Given February 3, due February 10.

Objective: To work with Taylor series in elementary numerical analysis.

1. Examine curve A and curve B. One of the functions has a jump in its third derivative while the other is completely smooth. Try to determine which curve has a discontinuous third derivative and locate the discontinuity on the graph. Do not spend much of time doing this in a quantitative way, but do make a guess. In the past, there have been more wrong than correct answers.

2. Extrapolation means estimating the value of a function from known values on one side only. For example, suppose \( g(x) \) is a smooth function of \( x \) and we know the values \( g(0) \), \( g(h) \), \( g(2h) \), and so on. We wish to estimate, say, \( g(-h) \) or \( g(-2h) \) from the known values. One instance would be the estimate:

\[
g(-2h) \approx a_0 g(0) + a_1 g(h) + a_2 g(2h) + a_3 g(3h) .
\] (1)

The coefficients happen not to depend on \( h \) in this problem. Show that we get first order accurate estimates of \( g(-h) \) or \( g(-2h) \) using only \( g(0) \), second order accuracy with \( g(0) \), and \( g(h) \), and so on to fourth order accuracy with (1). We say that \( g \) is constant if \( g(x) = b_0 \) for all \( x \), linear if \( g(x) = b_0 + b_1 x \), quadratic if \( g(x) = b_0 + b_1 x + b_2 x^2 \), etc. Show that these extrapolations are exact for constants, linears, quadratics, and cubics respectively.

3. The interval \((0, L)\) is divided into \( n \) subintervals, called cells, of equal length \( \Delta x = L/n \). Use the notation \( x_k = k \Delta x \) for the endpoints of the cells, and \( I_k = (x_k, x_{k+1}) \) for the cells themselves. It will be convenient to write \( x_{k+1/2} = (k + 1/2) \Delta x \) for the midpoint of \( I_k \). We do not have the values of \( f \), but we do have the cell averages

\[
F_k = \frac{1}{\Delta x} \int_{x_k}^{x_{k+1}} f(x) dx .
\]

Assume that \( f(x) \) is a smooth function. Show that \( F_k = f(x_{k+1/2}) + O(\Delta x^2) \). Hint: it will be easier if you use a variable \( y = x - x_{k+1/2} \) and perform Taylor series expansions about \( x = x_{k+1/2}, y = 0 \).

4. Find a second order accurate estimate of \( f(x_k) \) in terms of cell averages. The estimate will be different for the interior points, \( x_1, \ldots, x_{n-1} \) and the boundary points \( x_0 \) and \( x_n \). For the interior points, find an approximation

\[
f(x_k) = a F_{k-1} + b F_k + O(\Delta x^2) .
\]

The estimate of \( f(0) \) should use \( F_0 \) and \( F_1 \).
5. If we wish to use the interior estimation formula at \( x = 0 \), which would be \( f(0) \approx aF_{-1} + bF_0 \), we face the problem that \( F_{-1} \) is not known. We can circumvent this problem by extrapolating a value for \( F_{-1} \) using known values values \( F_0, F_1, \) etc. as in part 2. For example, suppose we use \( F_0, F_1, \) and \( F_2 \) to calculate \( \hat{F}_{-1} \approx F_{-1} \), then \( f(0) \approx \hat{f}_0 = a\hat{F}_{-1} + bF_0 \), the result will be an approximation \( \hat{f}_0 = cF_0 + dF_1 + eF_2 \). What order of extrapolation do you need to get a second order estimate of \( f(0) \)? Is this estimate the same as the second order estimate from part 4?

6. Find a fourth order estimate of \( f(x_{k+1/2}) \) using \( F \) values. First find the interior formula

\[
f(x_{k+1/2}) \approx aF_{k-1} + bF_k + cF_{k+1},
\]

that works when \( k > 0 \) and \( k < n \). Then, (as time permits) figure out what order of extrapolation and how many extrapolated values you need to achieve fourth order at the end points.