Assignment 4, due end of finals week

Corrections: [none yet.]

1. Consider the usual tridiagonal matrix A so that $Au = f$ is the standard discretization of the differential equation

$$\partial_x^2 u = f \ , \quad 0 < x < 1$$
$$u(0) = 0 \ , \quad u(1) = 0 \ .$$

Suppose the problem is discretized with $n-1$ interior points and uniform grid spacing $\Delta x = \frac{1}{n}$. Write a code to implement the basic conjugate gradient algorithm to solve the discrete problem and plot the 2−norm of the error (or the residual, or both) as a function of iteration number k. Start with initial condition $u_0 \neq 0$. Take $f = 0$, so the solution is $u = 0$. The plot should say n in the title, in a form something like "n is $\cdots$", and contain whatever other information is necessary to understand the plot (e.g., which u_0). Consider log-log plots to look for power laws and regular (linear-linear) plots to visualize initial transients. Experiment with initial conditions smooth and rough. Some possibilities to choose from and expand on are: $u_0(x) = x(1-x)$, or $u_0 = \delta(x - x_0)$ (a function that differs from 0 at a single grid point), or the values of u_0 being independent mean zero random numbers. Take the time to write a code that is so automated that it is easy to do sophisticated numerical experiments quickly.

   (a) The 2−norm condition number satisfies $\kappa = \text{cond}_2(A) \sim Cn^2$ for large n. Find a formula for C.
   Hint: Early in the semester we showed that the eigenvectors of A have the form $v_j = \sin(\xi j)$.

   (b) Compare the computed error to theoretical the theoretical error bound involving κ. Investigate how the discrepancy depends on n, on k, and on the initial condition. State some conclusions or conjectures based on your observations.

   (c) Experiment with the problem in 3D. Let n be the size of the vector u, which is $n = (\Delta x^{-1} - 1)^3$, assuming $\Delta x = \Delta y = \Delta z$ and each continuous space variable goes from 0 to 1. How does κ depend on n (power and constant)? Compare the upper bound to actual performance.

2. This exercise asks you to solve a Laplace equation to solve a simple model of fluid flow – irrotational, incompressible, constant density, steady flow

in two dimensions. The flow happens in a contrived geometry. A more realistic geometry would involve issues of meshing that would make it hard to do much meaningful computation is a the time available for a homework exercise. This problem does illustrate many features of more serious problems.

Consider a two dimensional steady fluid flow, with fluid velocity vector field $u(x,y) = (u_x(x,y), u_y(x,y))$. A *stream function* is a single function $\psi(x,y)$ that determines the velocity by

$$u_x(x,y) = \partial_y \psi(x,y) \ , \ \ u_y(x,y) = -\partial_x \psi(x,y) \ .$$

From vector calculus, we know that there is a stream function if the divergence of the velocity vector field is zero.

$$0 = \mathrm{div}(u)(x,y) = \partial_x u_x(x,y) + \partial_y u_y(x,y) \ .$$

See below for some explanation. The *vorticity*, denoted by $\omega(x,y)$, is the curl of the velocity field:

$$\omega(x,y) = \mathrm{curl}(u)(x,y) = \partial_x u_y(x,y) - \partial_y u_x(x,y) \ .$$

In simple models of flow, if the vorticity starts off being zero, it remains zero. A fluid with $\omega(x,y) \equiv 0$ is *irrotational*. A calculation (do it) shows that if a flow is both irrotational and incompressible (divergence free, $\mathrm{div}(u) \equiv 0$), then $\triangle\psi = \partial_x^2\psi + \partial_y^2\psi = 0$.

The stream function determines how much fluid is flowing between points (x_0, y_0) and (x_1, y_1). Let Γ be a curve in the plane that goes from one of the two points to the other. More precisely, Γ is the curve $(x(s), y(s))$ with $(x(0), y(0)) = (x_0, y_0)$ and $(x(1), y(1)) = (x_1, y_1)$. The rate of fluid flow across Γ is

$$\text{rate} = \int_{s=0}^{s=1} (u_x dy - u_y dx) \ .$$

Using vector calculus (google "Green's theorem") we can see that

$$\text{rate} = \psi(x_1, y_1) - \psi(x_0, y_0) \ .$$

In particular, any curve that goes from (x_0, y_0) to (x_1, y_1) has the same flow rate.

A theorem of fluid mechanics called *Bernoulli's law* (really, a theorem of vector calculus with hypotheses related to fluid models) states that the pressure field is given by

$$p(x,y) = p_0 - \tfrac{1}{2} \left\| u(x,y) \right\|^2 = p_0 - \tfrac{1}{2} \left\| \nabla\psi(x,y) \right\|^2 \ .$$

If you haven't studied fluid mechanics, you might be surprised at the consequence that the pressure is lower with the flow speed is higher. That was the original motivation for the shape of airplane wings – flat on the bottom and curved on top to make the air on top go faster.

Imagine a 2D fluid flowing in a region inside the unit square but outside a smaller square inside. The unit square (the "big" square) is $[0,1] \times [0,1]$. The smaller square is $[a,b]\times[a,b]$, a square with corners at (a,a) and (b,b), with $0 < a < b < 1$. The small square is aligned with the big square. The computational domain is

$$\Omega = \{(x,y) \mid (x,y) \in [0,1] \times [0,1]\ ,\ \ (x,y) \notin [a,b] \times [a,b]\}\ .$$

The boundary of Ω consists of two boundary components, the outer boundary and the inner boundary

$$\begin{aligned} \Gamma_o &= \{x = 0 \text{ or } x = 1 \text{ or } y = 0 \text{ or } y = 1\} \\ \Gamma_i &= \{x = a \text{ or } x = b \text{ or } y = a \text{ or } y = b\ \}\ . \end{aligned}$$

Assume that no fluid crosses either boundary component. This implies that the stream function takes one constant value on the outer boundary and a different constant value on the inner boundary. Since this problem is linear (fluid mechanics problems are almost never linear), it does not matter what the two boundary stream function values are, as long as they are different. Therefore, take $\psi(x,y) = 0$ on the outer boundary and $\psi(x,y) = 1$ on the inner boundary.

Write a code to compute the stream function and the pressure using the standard five point discretization of the Laplace operator. Choose $a = .25$, $b = .5$ and other values you are interested in. Choose $\Delta x = \Delta y$ and pick them so that there are mesh points on the inner and outer boundary. Make contour plots of the stream function and the pressure. Make sure to include information about the run in the plot title. Automate your code so that it is easy to experiment with. Solve the discrete equations using conjugate gradients. If you have time and interest, experiment with SSOR preconditioned conjugate gradients (not very complicated to code) or fancier methods (which take longer to code).

Most big computations have a *quantity of interest*, which is some function of the solution you are interested in. The quantity of interest for this calculation is the total pressure force on the bottom of the inner square

$$Q = \int_a^b p(x,a)\,dx\ .$$

To calculate this, you have to *post-process* the computed stream function to evaluate the pressure (Bernoulli's law, discretized in a sensible way) and approximate the integral (midpoint rule, trapezoid rule or whatever). Solve the discrete ψ equations accurately enough that more conjugate gradient iterations would not have a significant impact on the Q you get.

Comment on the results. Pay attention to these points

(a) What is the order of accuracy of Q as a function of Δx?

(b) Are the computed stream function and pressure fields smooth, or do they have singularities that could effect the accuracy of Q?

(c) How many conjugate gradient iterations does it take to evaluate Q? How badly would you like a solution strategy that takes fewer iterations?

A note on incompressibility and the material derivative

A velocity field with zero divergence is called *incompressible*. This comes from the density equation

$$\partial_t \rho + \partial_x(\rho u_x) + \partial_y(\rho u_y) \; .$$

The rate of change of the density of a piece of fluid carries with the velocity field is called the *material derivative* and written D_t. For any quantity $q(x, y, t)$, the material derivative is the rate of change of q if you "move with the fluid" which is the last two differential equations in:

$$D_t q = \frac{d}{dt} q(x(t), y(t), t) \; , \quad \frac{dx}{dt} = u_x(x, y, t) \; , \quad \frac{dy}{dt} = u_y(x, y, t) \; .$$

The chain rule gives the material derivative of ρ as

$$D_t \rho = \partial_t \rho + u_x \partial_x \rho + u_y \partial_y \rho \; .$$

Given the mass conservation equation above, if the material derivative of the density is zero, then the divergence of the velocity is zero (check this). Compressing a fluid means increasing its density, and decompressing means decreasing the density. A flow can be incompressible without having constant density. This could happen if some air is hotter (less dense) than other air, for example. In this situation, the density at a specific point can change in time, which is $\partial_t \rho(x, y, t) \neq 0$. This can happen, for example, if the wind blows hot air to where the air used to be cold. The fluid does not have to be compressed for this to happen.