Probability Limit Theorems, II, Homework

1. Is it possible to have a continuous time continuous path martingale $X(t)$ so that $X(0) \equiv 0$ but $\mathrm{E}[X(T)^2] = \infty$ for some $T < \infty$?

2. Let $f(t) = \sum_n \hat{f}_n e^{int}$, so that

$$\int_0^{2\pi} |f(t)|^2 \, dt = 2\pi \sum_n \left| \hat{f}_n \right|^2 < \infty \ .$$

Suppose independent random variables s_n have values $s_n = \pm 1$ and $\mathrm{E}(s_n) = 0$. Define

$$g(t) = \sum_n s_n \hat{f}_n e^{int} \ .$$

Let p be in the range $2 \leq p < \infty$. Show that $g \in L^p$ with probability one. Note that there is no reason for f to be in any L^p with $p > 2$. Hint: try p of the form $2k$.

3. A discrete time stochastic process, X_n, is a renewal process if there is an increasing family of stopping times, $\tau_1 < \tau_2 < \cdots$ with $\mathrm{E}[\tau_{k+1} - \tau_k \mid \mathcal{F}_{\tau_\|}] \leq \infty$ almost surely so that the process "starts over" at each τ_k. We can give the technical definition in terms of the k^{th} restarted process:

$$Y_n^k = X_{\tau_k + n} \ .$$

The condition is that the joint distribution of any finite part: $X_0, X_1, \cdots, X_N$ is the same as the joint distribution of $Y_0^k, \cdots, Y_N^k$, for each k and N, and that these finite parts are independent. Suppose that $\mathcal{S}$ is a finite or countable "state space" and that S_n is a positive recurrent indecomposable Markov chain on $\mathcal{S}$. Let $X_n = f(S_n)$ for some function, f.

a. If $S_0 = s^*$ is deterministic, show that X_n is a renewal process. Hint: look at the times when $S_n = s^*$.

b. Use this to show that

$$\bar{f} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n} f(S_k) \tag{1}$$

exists, at least if f is bounded. Hint: break the sum into sums over independent epochs (times between consecutive renewals). This gives a proof of the ergodic theorem for discrete space Markov chains that is independent of the ergodic theorem.

c. Under the stronger assumption that $\mathrm{E}[\tau_1^2] < \infty$, prove a central limit theorem for sums of the form (1).