Complex Variables II
Final Exam

Exam rules

- You may use the textbook by Marsden and Hoffman, the *Complex Analysis* book by Alfors, and your class notes. You may not discuss with anyone or use internet or other resources.
- The exam is due, in hardcopy in my mailbox (behind the guard station in the lobby of Warren Weaver Hall) by 9pm on Tuesday, May 16 (the last hour of finals week).
- Do not post questions about the exam on the NYU Classes discussion site for this class. Please email me immediately with questions. I will answer either individually or with a broadcast announcement to the class, whichever is appropriate.

Exam exercises

1. Suppose $f(z)$ is defined and analytic for all $\mathrm{Re}(z) > -1$ except for a simple pole at $z = 0$. Suppose f satisfies the inequality, for all real $x > 0$ and all real y,
$$\left| f(x) f(x+iy)^2 f(x+2iy) \right| \geq 1 \; .$$
Prove that $f(iy) \neq 0$ for all real y.

2. Use the Poisson summation formula to evaluate $S(w)$. Assume $\mathrm{Re}(w) > 0$.
$$S(w) = \sum_{n=1}^{\infty} \frac{1}{1 + wn^2} \; .$$
Show that $S(w)$ is an analytic function of w in this range. Does S have an analytic extension to a larger part of $\mathbb{C}$? If so, what is the largest such region? Use the behavior of $S(w)$ as $w \to \infty$ (in a suitable way) to evaluate
$$A = \sum_{n=1}^{\infty} \frac{1}{n^4} \; .$$
[*Explanation.* The Laurant series for $S(w)$ about $w = \infty$ has coefficients that involve what are called *Bernoulli* numbers, B_k. A longer version of this exercise would explore B_k for larger k.]

3. Show that if $u(x,y)$ is harmonic in the whole plane $\mathbb{R}^2$ and if there is a C with

$$|u(x,y)| \le C\left(1 + x^2 + y^2\right) \ ,$$

Then there are numbers $\alpha, \cdots, \epsilon$ so that

$$u(x,y) = \alpha\left(x^2 - y^2\right) + \beta xy + \gamma x + \delta y + \epsilon \ .$$

4. Prove the identity

$$0 = \int_{|z|=1} \log(|z-1|)\,|dz| = \int_0^{2\pi} \log\left(\left|e^{i\theta} - 1\right|\right) d\theta$$

The integral on the right is given in two equivalent ways for clarity. The exercise is to show the value is zero. *Hint.* Show that 0 is the value at the origin of a harmonic function (find the function). First ignore the fact that the integrand is not bounded, then fix that.

5. Suppose x_k is a strictly increasing (i.e., $x_{k+1} > x_k$) sequence of real numbers with

$$\sum_{k=1}^{\infty} \frac{1}{x_k} < \infty \ .$$

Consider the product

$$f(z) = \prod_{k=1}^{\infty}\left(1 - \frac{z}{x_k}\right) \ .$$

   (a) Show that the product defines an entire function of z.

   (b) Justify the formula

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^{\infty} \frac{1}{z - x_k} \ .$$

   (c) Show that if $f'(z) = 0$ then z is real.

   (d) Show that the zeros of f' *interlace* the zeros of f. That is, let $y_1 \le y_2 \le \cdots$ be all the zeros of f'. Show that $x_1 < y_1 < x_2 < y_2 < \cdots$ (all inequalities strict).

Explanation. Parts (b), (c) and (d) would be clear by simple calculations and graphing if f were a (finite degree) polynomial $p(z) = C(z-x_1)\cdots(z-x_n)$.

6. Consider the integral

$$I(n) = \int_0^1 \cos(n\pi(x + x^3))\,dx \ .$$

Show that $I(n)$ is an exponentially decreasing function of n, i.e., there are positive a and b with $|I_n| \le ae^{-bn}$. *Extra credit*: Show that the largest possible b is $\frac{\pi}{\sqrt{3}}$. *More extra credit*: Find c so that $I_n = \frac{c}{\sqrt{n}}e^{-\frac{\pi n}{\sqrt{3}}}\left(1 + o(\frac{1}{n}\right)$.