Complex Variables II
Assignment 9

1. The *Fourier transform* of a function $f(x)$ is another function $\widehat{f}(\xi)$ given by the *Fourier integral*[1]

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i\xi x} f(x)\, dx \; . \tag{1}$$

The *Fourier inversion formula* is

$$f(x) = \int_{-\infty}^{\infty} e^{2\pi i\xi x} \widehat{f}(\xi)\, d\xi \; . \tag{2}$$

This is the *Fourier representation* or the *plane wave* representation of f as a "sum" (integral) of complex exponentials $e^{i\xi x}$ with weights $\widehat{f}(\xi)$. The Fourier transform has many uses in pure and applied mathematics. Most math grad students are expected to know it, though there is no specific class that is required to cover it. Here is an approach that uses complex analysis. There are other approaches that do not require complex analysis. The method of this Exercise and Exercise 7 is a combination of the method of Exercise 17 (page 124 of *A Course of Modern Analysis* by E. T. Whittaker and G. N. Watson) and the method of Exercise 2 of Assignment 4.

The Fourier transform formula (1) and the Fourier inversion formula (2) differ only in the minus sigh in the exponent. Therefore, a derivation of property B about f from property A about $\widehat{f}$ usually also is a derivation of the property A about $\widehat{f}$ from property B from f. For example, if $\left|\widehat{f}(\xi)\right| \leq C\left(1+\xi^2\right)$ then $f(x)$ is continuous (check this but do not hand it in). Here "property A" is the inequality and "property B" is continuity. Similarly, if $|f(x)| \leq C\left(1+x^2\right)$, then $\widehat{f}(\xi)$ is a continuous function of ξ. The roles of the "space variable" x and the "frequency variable" ξ may be reversed in most Fourier formulas. This is useful, for example, in the Dirac formula (6).

This Exercise handles the Fourier formulas (1) and (2) under the hypothesis that f is analytic and exponentially decaying in a "strip" which will be a fixed width neighborhood of the x axis:

$$S_r = \{z \mid -r \leq \mathrm{Im}(z) \leq r\} \; .$$

[1] There are different conventions for the definition of the Fourier transform. The difference is usually in the location of the 2π factors. For example, physicists usually replace $2\pi\xi$ with ξ in (1) and (2). This requires one of them to get a pre-factor of $\frac{1}{2\pi}$.

We assume that the function $f(x)$ extends to an analytic function in some S_r and has exponential decay in the sense that there are constants $C > 0$ and $m > 0$ with

$$|f(z)| \leq Ce^{-m|x|} \; , \quad \text{for all } z = x + iy \in S_r \; . \tag{3}$$

(a) Show that $f(x) = e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ is in the function class (3). Find examples that decay only exponentially ($e^{-c|x|}$ rather than e^{-cx^2}) and only in a finite strip.

(b) Show that if f satisfies bounds of the form (3), then $\widehat{f}$ satisfies similar bounds (analytic in a strip around the x axis and uniform exponential decay in that strip), but possibly with different r and m.

(c) Show that if f is analytic in a neighborhood of S_r and satisfies bounds of the form (3), and if $|\text{Im}(z)| < r$, then $f(z) = f_+(z) + f_-(z)$ with

$$f_+(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(u - ir)}{u - z - ir} \, du$$
$$f_-(z) = \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(u + ir)}{u - z + is} \, du \; .$$

Show that f_+ is analytic in the upper half space $\text{Im}(z) > -r$, and f_- is analytic in the overlapping lower half space $\text{Im}(z) < r$. These satisfy $f_+(z) \to 0$ as $\text{Im}(z) \to \infty$ and $f_-(z) \to 0$ as $\text{Im}(z) \to -\infty$.

(d) Show that one of the following is true whenever $\text{Im}(w) \neq 0$. The one that is true depends on the sign of $\text{Im}(w)$

$$\int_0^{\infty} e^{-i\xi w} \, d\xi = \frac{1}{iw}$$
$$\int_{-\infty}^{0} e^{i\xi w} \, d\xi = -\frac{1}{iw} \; .$$

(e) Show that $\widehat{f}_+(\xi) = 0$ if $\xi < 0$ and $\widehat{f}_-(\xi) = 0$ for $\xi > 0$. You may omit details of the $\widehat{f}_-$ part, which is similar to the $\widehat{f}_+$ part. *Warning.* f_+ and f_- may not satisfy the inequalities (3) even if f does. The Fourier integrals may not converge absolutely.

(f) Show that

$$f_+(x) = \int_0^{\infty} e^{2\pi i\xi x} \widehat{f}_+(\xi) d\xi \; .$$

Sketch a similar fact about f_- and use this to verify the Fourier inversion formula (2). *Warning.* Parts (e) and (f) are hard. Please do not spend too much time on them.

2. Some standard examples of Fourier transforms are often given as exercises in contour integration. For each case, calculate $\widehat{(f)}(\xi)$ using (1) and verify the Fourier inversion formula (2) by explicit integration. These have appeared in earlier exercises, but now you will know why.

(a) $f(x) = e^{-x^2}$

(b) $f(x) = e^{-|x|}$. The calculation of $\widehat{f}$ does not require contour integration. In fact, f is not analytic, but does have exponential decay. The solution of Exercise 1(b) shows that the Fourier transform of a function with exponential decay is analytic in a strip, as this one is. The f here is not analytic and its Fourier transform does not have exponential decay.

3. Some basic properties of the Fourier transform follow directly from the definition.

   (a) (*scaling*) If $g(x) = f(\lambda x)$ then $\widehat{g}(\xi) = \frac{1}{\lambda}\widehat{f}(\xi/\lambda)$.

   (b) (*translation*) If $g(x) = f(x+a)$, then $\widehat{g}(\xi) = e^{-2\pi i \xi a}\widehat{f}(\xi)$.

   (c) (*differentiation*) If $g(x) = f'(x)$, then $\widehat{g}(\xi) = 2\pi i \xi \widehat{f}(\xi)$. (Most differential equations classes cover Fourier transforms for this reason.)

4. The *Direc delta "function"* is not a true function but it is an informal way to derive many facts about Fourier transforms. An informal definition of this informal function is

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{1}{2\epsilon}x^2} . \tag{4}$$

You might think that $\delta(x) = 0$ if $x \neq 0$, which is a consequence of the "definition", would imply that $\int \delta(x)dx = 0$. But no!. The definition is interpreted to imply that

$$\int_{-\infty}^{\infty} \delta(x)dx = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{1}{2\epsilon}x^2} dx = 1 . \tag{5}$$

Moreover, if $u(x)$ is a bounded and continuous function, then

$$\int_{-\infty}^{\infty} u(x)\delta(x)dx = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} u(x) \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{1}{2\epsilon}x^2} dx = u(0) .$$

This is interpreted as saying the delta function is a "point mass" at $x = 0$. If you think of $\delta(x)$ as a density, then the density is zero except at $x = 0$, and all the "mass" is concentrated at the point $x = 0$. The function $\delta(x - y)$ has all of its mass at the point y, in the sense that

$$\int_{-\infty}^{\infty} u(x)\delta(x - y)dx = u(y) .$$

This is a consequence of (5) after a change of variables.

Many basic Fourier formulas have informal derivations using the delta function and change of order of integration in multiple integrals. These derivations are not rigorous because some of the integrals involved do not converge, but they can be made rigorous using $\epsilon \to 0$ limits as in (5). This exercise asks you to do the informal, non-rigorous derivations to see how the world outside of strict academic mathematics uses the delta function.

The fundamental formula behind Fourier analysis is the informal Dirac formula

$$\int_{-\infty}^{\infty} e^{2\pi i\xi x} dx = \delta(\xi) \ . \tag{6}$$

This is not true in the simple sense that the integral on the left is equal to the function on the right for each ξ. The integral does not converge and $\delta(\xi)$ is not a function. It is true in a *regularized* sense hinted at with part (a). Here is an informal explanation of the Fourier inversion formula (2). The Dirac formula is used in the next to last step, with integration over ξ instead of x. The first steps are a change of the order of integration that is not justified because the integrals do not converge.

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{2\pi i\xi x} \widehat{f}(\xi) d\xi &= \int_{-\infty}^{\infty} e^{2\pi i\xi x} \left(\int_{-\infty}^{\infty} e^{-2\pi i\xi y} f(y) dy \right) d\xi \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i\xi(x-y)} f(y) dy d\xi \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-2\pi i\xi(x-y)} d\xi \right) f(y) dy \\
&= \int_{-\infty}^{\infty} \delta(x-y) f(y) dy \\
&= f(x) \ .
\end{aligned}$$

(a) We *regularize* the *plane wave* $e^{2\pi i\xi x}$ by multiplying by a slow "cut-off" function $\psi_\epsilon(x) = e^{-\frac{1}{2\epsilon}x^2}$. The regularized left side of (6) defines an "approximate delta function"

$$\int e^{2\pi i\xi x} \psi_\epsilon(x) dx = \delta_\epsilon(\xi) \ .$$

Evaluate $\delta_\epsilon(\xi)$ explicitly and show that it converges to $\delta(\xi)$ in the sense of (4).

(b) Derive the *Plancharel* relation

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} \left| \widehat{f}(\xi) \right|^2 d\xi \ .$$

Hint. Express $|f(x)|^2$ as a double integral

$$\overline{f}(x) f(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i\xi x} e^{2\pi i\eta x} \overline{\widehat{f}}(\xi) \widehat{f}(\eta) d\xi d\eta \ .$$

The left side of the Plancharel formula becomes a triple integral. Change order of integration (not rigorous, but do it anyway) and use the Dirac relation (6) to simplify.

(c) Derive the convolution formulas. If $h(x) = f(x)g(x)$, then

$$\widehat{h}(\zeta) = \int_{-\infty}^{\infty} \widehat{f}(\zeta - \xi)\widehat{g}(\xi) d\xi = \left(\widehat{f} * \widehat{g}\right)(\zeta) \; .$$

The $*$ on the right represents the *convolution integral* in the middle. Show that if $h = f * g$, then $\widehat{h} = \widehat{f}\widehat{g}$.

5. The *Poisson summation formula* is

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{-\infty}^{\infty} \widehat{f}(n) \quad \text{(sum over } n \in \mathbb{Z}) \; . \tag{7}$$

(a) Prove the Poisson summation formula (7) under the hypothesis that f satisfies the bounds (3). *Hint.* Show that the left side is

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x - is)}{e^{2\pi i (x - is)} - 1} dx - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x + is)}{e^{2\pi i (x + is)} - 1} dx \; .$$

Then get the right side by expressing the fractions as convergent power series.

(b) The *trapezoid rule* is the approximation of the integral by a Riemann sum

$$\int f(x) dx \approx \sum f(x_k) \Delta x \; .$$

Suppose $x_k = k\Delta x$ and f is analytic in S_r and satisfies (3). The error in the trapezoid rule for the integral over the whole line is

$$R_{\Delta x} = \sum_{-\infty}^{\infty} f(x_k)\Delta x - \int_{-\infty}^{\infty} f(x) dx \; .$$

Show that the trapezoid rule is *exponentially accurate* in the sense that there are constants C and $m > 0$ so that

$$|R_{\Delta x}| \leq C e^{-\frac{m}{\Delta x}} \; .$$

Hint. The integral is $\widehat{f}(0)$. Interpret the sum using the scaling formula from Exercise 3.

(c) The *Jacobi inversion formula* a famous application of the Poisson summation formula, but it is postponed to Assignment 10.